Affine Mappings and Related Conics

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Abstract

Two points in the plane of the reference triangle give several interesting conics related to these points. Several remarkable triangle points can be deduced of this basic configuration.

1 Preliminary definitions and notations

1.1 Generalities

Let ABC be the usual reference triangle and let $P = p : q : r$, $U = u : v : w$ (barycentric coordinates) be two points not lying on a sideline of ABC. All throughout the paper, the notation $\sum$ always denotes a cyclic sum.

Denote by :

- $A_pB_pC_p$ and $A_uB_uC_u$ the cevian triangles of P and U.
- $P_aP_bP_c$ and $U_aU_bU_c$ the anticevian triangles of P and U.
- $\mathcal{T}(P)$ the tripolar (trilinear polar) of P with equation $\sum \dfrac{x}{p} = 0$.
- $\mathcal{C}(P)$ the circum-conic with perspector P and equation $\sum \dfrac{p}{x} = 0$.
- $\mathcal{I}(P)$ the inscribed conic (inconic) with perspector P and equation

$$\sum \left(\frac{x}{p}\right)^2 - 2\left(\frac{y}{q}\right)\left(\frac{z}{r}\right) = 0.$$

  The center of $\mathcal{I}(P)$ is the complement of the isotomic conjugate of P. This is the point $p(q + r) : : .$

- $\mathcal{C}(P, U)$ the circum-conic passing through P, U with equation

$$\sum pu(rv - qw)yz = 0 \iff \sum \left(\frac{1}{rv} - \frac{1}{qw}\right) yz = 0.$$

  $\mathcal{C}(P, U)$ degenerates if and only if U lies on a cevian of P.

1.2 Isoconjugation

We briefly recall several known facts.

- Let $\mathcal{F}(P)$ be the pencil of diagonal conics passing through P and the vertices of its anticevian triangle $P_aP_bP_c$.

  The polar lines of U in all the conics of $\mathcal{F}(P)$ concur at a point called the $P-$isoconjugate of U which will be hereby denoted by $P(U)$.

  Obviously,

– $P(U) = P_a(U) = P_b(U) = P_c(U)$,

– the $P-$isoconjugates of P, P_a, P_b, P_c are themselves (they are fixed points),

– the $P-$isoconjugates of A, B, C are not defined (they are singular points).

- The isotomic conjugate tP of P is the $G-$isoconjugate of P (G is the centroid of ABC).
- The pole of the isoconjugation is the $P-$isoconjugate of G and is the (barycentric) square P^2 of P.

  P^2 is the intersection of these two lines :

  – the line through G and tc$P$ (the isotomic conjugate of the complement of P),

  – the line through P and ct$P$ (the complement of the isotomic conjugate of P).
- The (barycentric) cube P^3 of P is the $P-$isoconjugate of tP.

  P^3 is the intersection of these two lines :

  – the tangent at P to the circum-conic passing through P and tP,

  – the tangent at tP to the diagonal conic passing through P and tP.
- The (barycentric) product $P \times U$ of P and U is the pole of the isoconjugation which swaps them.

1.3 Special diagonal conics

- $\mathcal{D}(P, U)$ is the diagonal conic passing through P, U with equation

$$\sum (rv - qw)(rv + qw)x^2 = 0.$$

  Its center has first barycentric $\dfrac{1}{(rv + qw)(rv - qw)}$.

  $\mathcal{D}(P, U)$ contains P_a, P_b, P_c, U_a, U_b, U_c and more generally, the vertices of the anticevian triangle of any of its points.

  $\mathcal{D}(P, U)$ degenerates if and only if U lies on a cevian or an anticevian of P.
- $\Delta(P, U)$ is the diagonal conic passing through P, P_a, P_b, P_c which is tangent at P to the line PU, at P_i to the line P_iU_i ($i = a, b, c$). It has equation

$$\sum qr(rv - qw)x^2 = 0.$$

  Its center has first barycentric $\dfrac{p}{rv - qw}$.

  Similarly, $\Delta(U, P)$ is the diagonal conic passing through U, U_a, U_b, U_c which is tangent at U to the line PU, at U_i to the line P_iU_i ($i = a, b, c$). It has equation

$$\sum vw(rv - qw)x^2 = 0.$$

  Its center has first barycentric $\dfrac{u}{rv - qw}$.

  Note that the line passing these two latter centers is the tripolar of the point with first barycentric $\dfrac{1}{(rv - qw)^2}$.

  Remark : $\Delta(P, U)$ and $\Delta(U, P)$ have four common points which are the fixed points of the isoconjugation which swaps P and U. In other words, these are the square roots of the barycentric product $P \times U$.

2 Operations on points

In this section, we define several operations on points. Most of them are very common and already well documented, see [2] for example. Each box contains only the first barycentric coordinate of a point.

2.1 $\boxed{P \otimes U = \dfrac{1}{rv + qw}}$ cevian product of P and U

The pole of the line PU in $\mathcal{D}(P, U)$ is the Ceva-point of P and U in [2] or the cevian product of P and U.

$P \otimes U$ is the intersection of the lines $U, P(U)$ and $P, U(P)$.

The tripolar of this point $P \otimes U$ is the line denoted $\mathcal{L}(P, U)$ with equation

$$\sum (rv + qw)x = 0$$

to be compared with the equation of PU

$$\sum (rv - qw)x = 0.$$

Note that $\mathcal{L}(P, U)$ is also the polar of P in $\mathcal{C}(U)$ and the polar of U in $\mathcal{C}(P)$. This is also the mixed polar of P and U in the degenerate cubic which is the union of the sidelines of ABC.

Consider now the degenerate conic which is the union of the lines PU and $\mathcal{L}(P, U)$. Its equation takes the form

$$\sum (rv - qw)(rv + qw)x^2 - 2\sum pu(rv - qw)yz = 0$$

which shows that it belongs to the pencil of conics generated by $\mathcal{C}(P, U)$ and $\mathcal{D}(P, U)$. Hence, apart P and U, $\mathcal{C}(P, U)$ and $\mathcal{D}(P, U)$ have two (real or not) common points Q_1, Q_2 lying on this line $\mathcal{L}(P, U)$.

2.2 $\boxed{P \oplus U = \dfrac{1}{rv} + \dfrac{1}{qw}}$ crosspoint of P and U

The pole of the line PU in $\mathcal{C}(P, U)$ is the crosspoint of P and U in [2] which we denote $P \oplus U$.

$P \oplus U$ is the intersection of the lines $P, P(U)$ and $U, U(P)$.

2.3 $\boxed{P \boxtimes U = \dfrac{1}{rv - qw}}$ tripole of the line PU

This is the fourth common point of the circum-conics $\mathcal{C}(P)$, $\mathcal{C}(U)$.

2.4 $\boxed{P \boxplus U = \dfrac{1}{rv} - \dfrac{1}{qw}}$ intersection of the tripolars of P, U

This is the perspector of the circum-conic $\mathcal{C}(P, U)$: the tangents at A, B, C to $\mathcal{C}(P, U)$ bound the (tangential) triangle $T_aT_bT_c$ which is perspective at $P \boxplus U$ to ABC.

2.5 $\boxed{P \oslash U = u\left(-\frac{u}{p}+\frac{v}{q}+\frac{w}{r}\right)}$ **cevian quotient of P and U**

The pole of $\mathcal{T}(P)$ in $\mathcal{C}(U)$ is the $P-$Ceva conjugate of U in [2] or cevian quotient of P and U. $P \oslash U$ solves the equation $X \otimes U = P$ i.e. $X \otimes U = P \iff X = P \oslash U$.

$P \oslash U$ is the perspector of the triangles $A_pB_pC_p$ and $U_aU_bU_c$.

Remarks :

– the lines $U, P \oslash U$ and $P, U \oslash P$ are the tangents at U and P to the conic $\mathcal{C}(P, U)$. These tangents obviously meet at $P \oplus U$. In particular, $G \oslash U$ is the center of $\mathcal{C}(U)$ since it is the pole of the line at infinity in the conic.

– $P \oslash U$ lies on $\Delta(U, P)$.

– P^2 lies on the line $P, G \oslash P$.

2.6 $\boxed{P \ominus U = \dfrac{u}{-\frac{p}{u}+\frac{q}{v}+\frac{r}{w}}}$ $P-$**crossconjugate of U**

$P \ominus U$ is the $P-$crossconjugate of U in [2] where it is defined as the perspector of the triangle ABC and the triangle with vertices $PA_u \cap B_uC_u$, $PB_u \cap C_uA_u$, $PC_u \cap A_uB_u$.

$P \ominus U$ solves the equation $X \oplus U = P$ i.e. $X \oplus U = P \iff X = P \ominus U$.

2.7 $\boxed{P\text{ⓐ}U = p\left(-\frac{u}{p}+\frac{v}{q}+\frac{w}{r}\right)}$ $P-$**anticomplement of U**

This is the image of U in the homology h_P with center P, axis $\mathcal{T}(P)$ which transforms the cevian triangle $A_pB_pC_p$ of P into ABC. This homology maps $\mathcal{I}(P)$ to $\mathcal{C}(P)$ and the center of $\mathcal{I}(P)$ to P^2.

This generalize the usual anticomplement obtained with $P = G$, the centroid of ABC, in which case the conics are the Steiner ellipses. When $P = K$ (Lemoine point), the conics are the Brocard ellipse and the circumcircle of ABC.

Note that the barycentric product of PⓐU and $U \ominus P$ is P^2 hence PⓐU is also the $P-$isoconjugate of $U \ominus P$.

2.8 $\boxed{P\text{ⓒ}U = p\left(\frac{v}{q}+\frac{w}{r}\right)}$ $P-$**complement of U**

This is the image of U in the reciprocal homology of h_P. In other words, Pⓐ$U = X \iff P$ⓒ$X = U$.

Note that P, U, PⓐU and PⓒU are collinear. More precisely, we see that PⓐU can be rewritten under the form Pⓐ$U = -u + p\left(\frac{v}{q}+\frac{w}{r}\right)$ which shows that PⓐU is the harmonic conjugate of P with respect to PⓒU and U.

PⓒU is also the $P-$isoconjugate of $P \otimes U$.

2.9 $\boxed{\mathbf{P\text{ⓗ}U} = qru^2 - vwp^2}$ $P-$**Hirst inverse of U**

Pⓗ$U = PU \cap \mathcal{L}(P, U)$ is the $P-$Hirst inverse of U in [2].

Recall that $PU \cup \mathcal{L}(P, U)$ is the (always real) decomposed conic of the pencil generated by $\mathcal{C}(P, U)$ and $\mathcal{D}(P, U)$.

PⓗU is also the intersection of the lines $P(U), U(P)$ and $P \oslash U, U \oslash P$.

The polar of PⓗU in $\mathcal{C}(P,U)$ or $\mathcal{D}(P,U)$ is the line with equation

$$\sum(rv-qw)(rv+qw)(qru^2-vwp^2)x=0$$

which contains $P\otimes U$, $P\oplus U$ and $P\boxtimes U$.

2.10 $\boxed{\mathbf{P}\text{ⓚ}\mathbf{U}=qru^2+vwp^2}$

PⓚU is either :

– the intersection of the lines $P, P\oslash U$ and $U, U\oslash P$ (compare with $P\oplus U$ which is the interection of $U, P\oslash U$ and $P, U\oslash P$),

– the harmonic conjugate of PⓗU with respect to $P(U)$ and $U(P)$,

– the harmonic conjugate of $P\oplus U$ with respect to $P\otimes U$ and $P\boxtimes U$.

2.11 Examples

The following table shows a small selection of these points where P and U are two very usual centers.

X^* is the isogonal conjugate of X.

Table 1: Examples of points

P	X_1	X_1	X_1	X_2	X_2
U	X_2	X_3	X_4	X_6	X_4
$P(U)$	X_6	X_4	X_3	X_{76}	X_{69}
$U(P)$	X_{75}	X_{255}	X_{158}	X_{32}	X_{393}
$P\otimes U$	X_{86}	X_{21}	X_{29}	X_{83}	X_{264}
$P\oplus U$	X_{37}	X_{73}	X_{65}	X_{39}	X_6
$P\boxtimes U$	X_{190}	X_{651}	X_{653}	X_{99}	X_{648}
$P\boxplus U$	X_{513}	X_{652}	X_{650}	X_{512}	X_{523}
$P\oslash U$	X_{192}	X_{3157}	X_{1148}	X_3	X_{1249}
$U\oslash P$	X_9	X_{1745}	X_{46}	X_{194}	X_{193}
$P\ominus U$	X_7	X^*_{1148}	X^*_{3157}	X_{3224}	X^*_{3167}
$U\ominus P$	X_{87}	X_4	X_{90}	X_4	X_{253}
PⓐU	X_{43}	X_{46}	X_{1745}	X_{69}	X_{20}
PⓒU	X_{42}	X_{65}	X_{73}	X_{141}	X_3
UⓐP	X_8	?	X_{1068}	X_{1613}	$1-a^2/S_A$
UⓒP	X_{10}	?	X_{225}	X_{3051}	X_{25}
PⓗU	X_{239}	X_{1936}	X_{243}	X_{385}	X_{297}
PⓚU	X_{894}	X_{1935}	X_{1940}	X_{384}	$a^2-1/(a^2S_A)$

References

[1] C. Kimberling, Triangle Centers and Central Triangles, *Congressus Numerantium*, 129 (1998) 1–295.

[2] C. Kimberling, *Encyclopedia of Triangle Centers*, http://www2.evansville.edu/ck6/encyclopedia.