Special Isocubics in the Triangle Plane

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This paper is organized into five main parts :

- a reminder of poles and polars with respect to a cubic.
- a study on central, oblique, axial isocubics i.e. invariant under a central, oblique, axial (orthogonal) symmetry followed by a generalization with harmonic homologies.
- a study on circular isocubics i.e. cubics passing through the circular points at infinity.
- a study on equilateral isocubics i.e. cubics denoted $\mathcal{K}_{60}$ with three real distinct asymptotes making 60° angles with one another.
- a study on conico-pivotal isocubics i.e. such that the line through two isoconjugate points envelopes a conic.

A number of practical constructions is provided and many examples of "unusual" cubics appear.

Most of these cubics (and many other) can be seen on the web-site :

http://bernard.gibert.pagesperso-orange.fr

where they are detailed and referenced under a catalogue number of the form **Knnn**.

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Chapter 1

Preliminaries and definitions

1.1 Notations

- We will denote by $\mathcal{K}$ the cubic curve with barycentric equation

$$F(x, y, z) = 0$$

  where F is a third degree homogeneous polynomial in x, y, z. Its partial derivatives will be noted F'_x for $\dfrac{\partial F}{\partial x}$ and F''_{xy} for $\dfrac{\partial^2 F}{\partial x \partial y}$ when no confusion is possible.

- Any cubic with three real distinct asymptotes making 60° angles with one another will be called an equilateral cubic or a $\mathcal{K}_{60}$. If, moreover, the three asymptotes are concurrent, it will be called a $\mathcal{K}_{60}^{+}$. At last, a $\mathcal{K}_{60}^{+}$ with asymptotes concurring on the curve is denoted by $\mathcal{K}_{60}^{++}$.

- The line at infinity is denoted by $\mathcal{L}^{\infty}$ with equation $x+y+z=0$. It is the trilinear polar of the centroid G. More generally, the trilinear polar of point P is denoted by $\mathbb{P}(P)$.

- Several usual transformations are very frequent in this paper. We will use the following notations :

  - $\mathbf{g}P$ = isogonal conjugate of P.
  - $\mathbf{t}P$ = isotomic conjugate of P.
  - $\mathbf{c}P$ = complement of P.
  - $\mathbf{a}P$ = anticomplement of P.
  - $\mathbf{i}P$ = inverse of P in the circumcircle.

  They easily combine between themselves and/or with other notations as in :

  - $\mathbf{tg}P$ = isotomic conjugate of isogonal conjugate of P.
  - $\mathbf{gig}P$ = isogonal conjugate of inverse (in the circumcircle) of isogonal conjugate of P = antigonal of P.
  - $\mathbf{a}X_{13}$, $\mathbb{P}(\mathbf{t}P)$, etc.

  The homothety with center P and ratio k is denoted by $h_{P,k}$.

- We will use J.H.Conway's notations :

$$S_A = \frac{1}{2}(b^2 + c^2 - a^2), S_B = \frac{1}{2}(c^2 + a^2 - b^2), S_C = \frac{1}{2}(a^2 + b^2 - c^2).$$

- The barycentric coordinates $(f(a, b, c) : f(b, c, a) : f(c, a, b))$ of a triangle center are shortened under the form $[f(a, b, c)]$.

- Usual triangle centers in triangle ABC [1] :

  - I = incenter $= X_1 = [a]$.
  - G = centroid $= X_2 = [1]$.
  - O = circumcenter $= X_3 = [a^2 S_A]$.
  - H = orthocenter $= X_4 = [1/S_A]$.
  - K = Lemoine point $= X_6 = [a^2]$.
  - L = de Longchamps point $= X_{20} = [(b^2 - c^2)^2 + 2a^2(b^2 + c^2) - 3a^4]$.
  - $X_{30} = [(b^2 - c^2)^2 + a^2(b^2 + c^2) - 2a^4]$ = point at infinity of the Euler line.
  - $\mathbf{t}K$ = third Brocard point $= X_{76} = [1/a^2]$.

- P/Q **: cevian quotient or Ceva-conjugate**

  Let P and Q be two points not lying on a sideline of triangle ABC. The cevian triangle of P and the anticevian triangle of Q are perspective at the point denoted by P/Q called cevian quotient of P and Q or $P-$Ceva-conjugate of Q in [38], p.57. Clearly, $P/(P/Q) = Q$.

  $P \star Q$ **: cevian product or Ceva-point**

  The cevian product (or Ceva-point in [39]) of P and Q is the point X such that $P = X/Q$ or $Q = X/P$. It is denoted by $P \star Q$. It is equivalently the trilinear pole of the polar of P (resp. Q) in the circum-conic with perspector Q (resp. P).

  If $P = (u_1 : v_1 : w_1)$ and $Q = (u_2 : v_2 : w_2)$, then

  $P/Q = u_2\left(-\frac{u_2}{u_1} + \frac{v_2}{v_1} + \frac{w_2}{w_1}\right) : :$ and $P \star Q = \frac{1}{v_1 w_2 + v_2 w_1} : : .$

1.2 Isoconjugation

1.2.1 Definitions

- Isoconjugation is a purely projective notion entirely defined with the knowledge of a pencil of conics such that triangle ABC is self-polar with respect to any conic of the pencil.

  For any point M – distinct of A, B, C – the polar lines of M with respect to all the conics of the pencil are concurrent at M^* (which is the pole of M in the pencil of conics).

  We call **isoconjugation** the mapping $\varphi\ : M \mapsto M^*$ and we say that M^* is the **isoconjugate** of M.

[1] We make use of Clark Kimberling's notations. See [38, 39].

φ is an involutive quadratic mapping with singular points A, B, C and fixed points the common points – real or not – of the members of the pencil. Since all the conics of this pencil have an equation of the form $\alpha x^2 + \beta y^2 + \gamma z^2 = 0$, they are said to be diagonal conics.

- If we know two distinct isoconjugate points P and $P^* = \varphi(P)$ – provided that they do not lie on ABC sidelines – we are able to construct the isoconjugate M^* of any point M.

  The (ruler alone) method is the following : [2]

  - let T be any point on the line PP^* (distinct of P and P^*) and $A_T B_T C_T$ the cevian triangle of T.
  - $A_1 = PA \cap P^*A_T, B_1 = PB \cap P^*B_T, C_1 = PC \cap P^*C_T$.
  - $A' = MA_1 \cap BC, B' = MB_1 \cap CA, C' = MC_1 \cap AB$.
  - $A'' = B_1C' \cap C_1B', B'' = C_1A' \cap A_1C', C'' = A_1B' \cap B_1A'$.
  - M^* is the perspector of triangles ABC and $A''B''C''$.

  Another more simple but less symmetric method is :

  - $E_0 = AM \cap CP^*$,
  - $E_1 = MP \cap BC$,
  - $Q = AP \cap E_0E_1$,
  - $E_2 = E_0Q \cap AB$,
  - $M^* = CQ \cap P^*E_2$.

  The knowledge of these two distinct isoconjugate points P and P^* is sufficient to obtain two members of the pencil of diagonal conics as defined in the paragraph above :

  – one is the conic $\gamma(P)$ through P and the vertices of the anticevian triangle of P which is tangent at P to the line PP^*. This conic also contains P^*/P and the vertices of its anticevian triangle.

  – the other is the conic $\gamma(P^*)$ through P^* and the vertices of the anticevian triangle of P^* which is tangent at P^* to the line PP^*. It also passes through $Q = P/P^*$ and the vertices of the anticevian triangle of Q.

- We now define the **pole** of the isoconjugation as the isoconjugate $\Omega = G^*$ of the centroid G. In other words, Ω is the intersection of the two polar lines of G in the two conics $\gamma(P)$ and $\gamma(P^*)$.

  This shows that there is no need of coordinates to define an isoconjugation. Nevertheless, since a lot of computation is needed for this paper, we will make use of barycentric coordinates and, if $\Omega = (p : q : r)$, the isoconjugation with pole Ω is the mapping :

$$\varphi_\Omega \ : M(x : y : z) \mapsto M^* \left(\frac{p}{x} : \frac{q}{y} : \frac{r}{z} \right) \sim (pyz : qzx : rxy)$$

[2] See [13] for details.

Remark :

Ω is the perspector of the circum-conic isoconjugate of $\mathcal{L}^\infty$. Thus, this conic is the locus of centers of all the diagonal conics of the pencil and, in particular, contains the centers of $\gamma(P)$ and $\gamma(P^*)$. It also passes through the six midpoints of the quadrilateral formed by the four fixed points of the isoconjugation. (See another construction below)

- When $\Omega = K$, we have the usual isogonal conjugation and when $\Omega = G$, we have the isotomic conjugation.
- When Ω is inside the triangle ABC i.e. p, q, r are all positive, the Ω-isoconjugation has four real fixed points : one is inside ABC and the three others are its harmonic associates, [3] but when Ω is outside the triangle ABC, the four fixed points are imaginary. The four fixed points are said to be the **square roots** of Ω and are denoted R_o, R_a, R_b, R_c where R_o is the one which is inside ABC when Ω is itself inside the triangle.

1.2.2 Useful constructions

More information and other constructions can be found in [60] and [13].

- **Barycentric product of two points** [4]

  The barycentric product $X \times Y$ of two distinct points X and Y is the pole of the isoconjugation which swaps them and therefore the line XY and the circum-conic through X and Y. [5]

  With $X = (u_1 : v_1 : w_1)$ and $Y = (u_2 : v_2 : w_2)$, we have $X \times Y = (u_1u_2 : v_1v_2 : w_1w_2)$ hence its name.

  If X and Y are distinct points, $X \times Y$ is the intersection of the polars of G in the two conics $\gamma(X)$ and $\gamma(Y)$ defined as above : $\gamma(X)$ passes through X and the vertices of the anticevian triangle of X and is tangent at X to the line XY. Note that $\gamma(X)$ also contains the cevian quotient Y/X.

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[3] The three harmonic associates of the point $M(\alpha : \beta : \gamma)$ are $(-\alpha : \beta : \gamma)$, $(\alpha : -\beta : \gamma)$, $(\alpha : \beta : -\gamma)$. They are the vertices of the anticevian triangle of M.

[4] See [61] for details and proofs.

[5] The pole of the line XY in this conic is called **crosspoint** of X and Y in [39].
If $X = (u_1 : v_1 : w_1)$ and $Y = (u_2 : v_2 : w_2)$, then this point has coordinates :

$$\left(\frac{1}{v_1w_2} + \frac{1}{v_2w_1} : \ : \ \right).$$

This has to be compared with the intersection of the trilinear polars of X and Y which is :

$$\left(\frac{1}{v_1w_2} - \frac{1}{v_2w_1} : \ : \ \right),$$

with the cevian product :

$$X \star Y = \left(\frac{1}{v_1w_2 + v_2w_1} : \ : \ \right),$$

and with the trilinear pole of the line XY which is :

$$\left(\frac{1}{v_1w_2 - v_2w_1} : \ : \ \right),$$

this latter point being the "fourth" intersection of the circum-conics with perspectors X and Y.

Another construction of $X \times Y$ (for distinct points) is the following :

Let X and Y be two points. X_a, X_b, X_c and Y_a, Y_b, Y_c are the vertices of the cevian triangles of X and Y resp. X_a^c on AB and X_a^b on AC are two points such that $AX_a^cX_aX_a^b$ is a parallelogram. Define Y_a^c on AB and Y_a^b on AC then intersect the lines $X_a^bY_a^c$ and $X_a^cY_a^b$ at Z_a. Define Z_b and Z_c similarly. The lines AZ_a, BZ_b, CZ_c concur at the requested point.

If U denotes the foot (on BC) of the radical axis of circles with diameters BC and X_aY_a, and if V, W are defined likewise on CA, AB respectively then U, V, W are collinear on a line whose trilinear pole is $X \times Y$. Practically, the circumcircle and the circle AX_aY_a meet again at A' and U is the intersection of BC and AA'.

- **Barycentric square of a point**

  The construction above is also valid when $X = Y$ since the circle AX_aY_a is now tangent at $X_a = Y_a$ to BC. This gives the barycentric square of point X denoted by X^2.
  This point X^2 is also :
  – the intersection of the lines G,$\mathbf{tc}X$ and X,$\mathbf{ct}X$,
  – the intersection of the polars of G in the pairs of lines passing through the vertices of the quadrilateral formed by X and its harmonic associates (vertices of the anticevian triangle of X),
  – the trilinear pole of the line passing through the midpoints of a vertex of the cevian triangle of X and the foot of $\mathbb{P}(X)$ on the same sideline of ABC.

  Remark :

  This construction can be used to construct the pole Ω of an isoconjugation since it is the barycentric product of two isoconjugate points P and P^*.

- **Ω-isoconjugate of a point X**

  The construction seen in §1.2.1 can be used but we can also define it as the barycentric product of the two points Ω and $\mathbf{t}X$.

- **Ω-isoconjugate of a line ℓ**

  Let X be the tripole of ℓ, X^* its Ω-isoconjugate, $E_X = G/X^*$.

  The Ω-isoconjugate of the line ℓ is the circumconic centered at E_X.

  This conic is inscribed in the anticevian triangle of X^*, the contacts being A, B, C.

  Remark :

  The construction of the Ω-isoconjugate of $\mathcal{L}^\infty$ is simpler : its center E is G/Ω, perspector of the medial triangle and the anticevian triangle of Ω. [6]

- **Square roots of Ω**

  We take Ω inside ABC to get real points.

  Let $\Omega_a, \Omega_b, \Omega_c$ and M_a, M_b, M_c be the vertices of the cevian triangles of Ω and G resp.

[6] The coordinates of E are $[p(q + r - p)]$.

The inversion swapping B and C, Ω_a and M_a has two fixed points on BC called F'_a and F''_a [7]. Define F'_b and F''_b, F'_c and F''_c similarly. These six points are three by three collinear on four lines whose tripoles are the four roots R_o, R_a, R_b, R_c we were looking for.

When the isoconjugation is defined by two distinct isoconjugate points P and Q, another construction of the square roots of the pole $\Omega = P \times Q$ may be construed as the intersection of the two conics γ_P and γ_Q where

– γ_P is the conic tangent at P to the line PQ passing through P and the vertices of the anticevian triangle of P,

– γ_Q is the conic tangent at Q to the line PQ passing through Q and the vertices of the anticevian triangle of Q.

Remarks :

1. The six lines passing through one vertex of ABC and the two fixed points on the opposite side intersect three by three at the same points R_o, R_a, R_b, R_c. The union of two lines through a vertex of ABC is in fact one of the degenerate conic of the pencil of conics seen in §1.2.1.
2. F'_a and F''_a can easily be obtained in the following manner : draw the perpendicular at Ω_a to BC intersecting the circle with diameter BC at two points, one denoted by Ω'_a. The bisectors of $\angle B\Omega'_a C$ intersect BC at F'_a and F''_a.

1.3 Isocubics

By an isocubic we mean a circum-cubic which is invariant under an isoconjugation. It is a known fact (see [5] for instance) that an isocubic $\mathcal{K}$ is invariant under an $\Omega(p : q : r)$-isoconjugation if and only if it has a barycentric equation of one of the two types :

$$(p\mathcal{K}) : \quad ux\,(ry^2 - qz^2) + vy\,(pz^2 - rx^2) + wz\,(qx^2 - py^2) = 0 \tag{1.1}$$

$$(n\mathcal{K}) : \quad ux\,(ry^2 + qz^2) + vy\,(pz^2 + rx^2) + wz\,(qx^2 + py^2) + k\,xyz = 0 \tag{1.2}$$

where u, v, w, k are real numbers whose signification is detailed in the following paragraphs.

Combining the notations above will lead to denote by $p\mathcal{K}_{60}$ a pivotal isocubic with three real distinct asymptotes making 60° angles with one another, by $n\mathcal{K}_{60}^{+}$ a non-pivotal isocubic with three real distinct **concurring** asymptotes making 60° angles with one another, and so on...

1.4 Pivotal isocubics or $p\mathcal{K}$

1.4.1 Definitions

A pivotal isocubic of the form $(p\mathcal{K})$ is the locus of M for which the points M, M^* and $P = (u : v : w)$ are collinear. For this reason, P is called the **pivot** of the isocubic. We shall simply refer to a pivotal isocubic as a $p\mathcal{K}$.

[7] F'_a and F''_a are the Poncelet points of the pencil of circles generated by the two circles with diameters BC and $\Omega_a M_a$: each circle of the pencil is orthogonal to the circle with diameter $F'_a F''_a$.

Remarks :

1. $p\mathcal{K}$ passes through A, B, C, P, P^*, the traces A_P, B_P, C_P of P, the square roots R_o, R_a, R_b, R_c of Ω.

2. This $p\mathcal{K}$ is equivalently the locus of point M such that :

   - the polar lines of M and M^* in the circum-conic centered at P are parallel.
   - P^*, M and P/M are collinear. For this reason, P^* is called **isopivot** or **secondary pivot** of the cubic. Note that P^* is the tangential of P.

3. This $p\mathcal{K}$ is also the locus of the contacts M, N of tangents drawn through P^* to the circum-conics passing through P. The line MN passes through P/P^* and P/P^*, M^*, $(P/M)^*$ are collinear. Naturally, P/P^*, M and $(P/M^*)^*$ are also collinear. Note that P/P^* is the tangential of P^*.

1.4.2 $p\mathcal{K}$ and trilinear polars

- The locus of point M such that the trilinear polars of M and its Ω-isoconjugate M^* are parallel is a $p\mathcal{K}$ with pivot Ω i.e. in this case pivot = pole. Its equation is :

$$\sum_{\text{cyclic}} px\,(ry^2 - qz^2) = 0.$$

  A $p\mathcal{K}$ with pivot = pole always contains the centroid (which is the isoconjugate of the pivot) and is always tangent at A, B, C to the medians. When $\Omega = G$, the cubic degenerates into the union of the three medians.

- More generally, given a line $\mathcal{L}$, the locus of point M such that the trilinear polars of M and its Ω-isoconjugate M^* concur on $\mathcal{L}$ is the $p\mathcal{K}$ with pole Ω and pivot the isoconjugate of the trilinear pole of the line $\mathcal{L}$.
  Conversely, any $p\mathcal{K}$ with pivot P can be seen as the locus of point M such that the trilinear polars of M, M^* and P^* are concurrent and, equivalently, the locus of point M such that the trilinear pole of the line MM^* lies on the trilinear polar of P^*.

- For example, the trilinear polars of any two isogonal conjugates on the Thomson and McCay cubics concur on the Lemoine axis and orthic axis respectively.

1.4.3 Construction of a $p\mathcal{K}$

Let P be the pivot and P^* its isoconjugate. [8] Let M be a variable point on the line PP^*. Draw $N = M/P$ perspector of the cevian triangle $A_M B_M C_M$ of M and the anticevian triangle $A^P B^P C^P$ of P. The circum-conic through M and P^* intersects the line PN at two points U and U^* which are isoconjugate points on the cubic and harmonic conjugates with respect to P and N.
The tangents to the cubic at these two points meet at N^*, which is also the second intersection of the same circum-conic (which passes through M, P^*, U, U^*) and the line MN.

[8] These two points define the isoconjugation.

Remarks and other constructions :

1. The locus of the perspector N is the conic through P and its harmonic associates [9] A^P, B^P, C^P, and also the (not necessarily real) square roots R_o, R_a, R_b, R_c of the pole Ω. This conic is tangent at P to the line PP^* which allows its construction. This is the polar conic $\mathcal{C}_P$ of P, see §2 and specially §2.4 below.

2. The isoconjugate M^* of M is the second intersection of PN with the circum-conic $\mathcal{C}_{P^*}$ through P and P^*, which is the polar conic of P^*.

3. The points P/U and P/U^* lie on the isocubic. The line joining these points passes through a fixed point Q of the curve with coordinates :

$$\left(\frac{u}{-\frac{p}{u^2}+\frac{q}{v^2}+\frac{r}{w^2}} : \frac{v}{\frac{p}{u^2}-\frac{q}{v^2}+\frac{r}{w^2}} : \frac{w}{\frac{p}{u^2}+\frac{q}{v^2}-\frac{r}{w^2}} \right).$$

   Q is the $\Omega-$isoconjugate of P/P^* or the P^*-cross conjugate of P.
   Conversely, for given Ω and P, the locus of U such that Q, P/U and P/U^* are collinear is the union of the $p\mathcal{K}$ and a $n\mathcal{K}$ with the same pole, with root Q (see §1.5 below).

4. **Tangentials of U and U^***

   The tangential of P is the point P^*. Denote by $\widetilde{U}$ and $\widetilde{U^*}$ the tangentials of U and U^*. Since U, U^*, P are collinear on the cubic, so are their tangentials $\widetilde{U}$, $\widetilde{U^*}$ and P^*. These tangentials can be constructed as

$$\widetilde{U} = U\,N^* \cap P/U^*\,(P/U)^* \quad \text{and} \quad \widetilde{U^*} = U^*\,N^* \cap P/U\,(P/U^*)^*.$$

   The isoconjugates $(\widetilde{U})^*$ and $(\widetilde{U^*})^*$ of these two tangentials are :

$$(\widetilde{U})^* = \widetilde{U}\,P \cap U^*\,P/U \quad \text{and} \quad (\widetilde{U^*})^* = \widetilde{U^*}\,P \cap U\,P/U^*$$

   and the points $P/P^*, (\widetilde{U})^*$ and $(\widetilde{U^*})^*$ are collinear on the cubic.

5. Remember that $A_PB_PC_P$ is the cevian triangle of P. The third point A_3 of the cubic on AU lies on the line through A_P and P/U^*. Similarly we obtain B_3, C_3 on the lines BU, CU respectively.

   In the same manner, the third point on AU^* is the isoconjugate A_3^* of A_3 on the line through A_P and P/U, B_3^* and C_3^* likewise.

   Note that the four points U, A_3^*, B_3^* and C_3^* share the same tangential $\widetilde{U}$. We say that these four points are the **pretangentials** of $\widetilde{U}$. Similarly U^*, A_3, B_3, C_3 are the pretangentials of $\widetilde{U^*}$.

6. **Polar conic of U**

   It is now possible to draw the polar conic $\mathcal{C}_U$ of U since it contains U and four other points harmonic conjugates of U with respect to A and A_3, B and B_3, C and C_3, P and U^*.

   The polar conic $\mathcal{C}_{U^*}$ of U^* is obtained likewise with the point U^* and its harmonic conjugates with respect to A and A_3^*, B and B_3^*, C and C_3^*, P and U.
   These two polar conics intersect at four points lying on the diagonal conic $\mathcal{C}_P$. These four points are the poles of the line UU^* in the cubic (see §2.3.1).

[9] Remember that the harmonic associates of a point are the vertices of its anticevian triangle.

7. **Pretangentials of U**

   The (not always real) pretangentials U_1, U_2, U_3, U_4 of U lie on $\mathcal{C}_U$: they are the contacts of the tangents drawn through U to the cubic other than the tangent at U itself.

   The two lines through B, $AU^* \cap CB_3^*$ and C, $AU^* \cap BC_3^*$ meet at a point which is the perspector of a conic inscribed in the triangle BCA_3. The two tangents to this conic passing through A_3^* contain U_1, U_2, U_3, U_4. Similarly we can draw two other pairs of tangents passing through B_3^* and C_3^*. These six tangents form a complete quadrilateral with vertices U_1, U_2, U_3, U_4 and diagonal triangle $A_3^*B_3^*C_3^*$.

   It follows that the cubic is a pivotal cubic with respect to $A_3^*B_3^*C_3^*$, with pivot U, isopivot $\widetilde{U}$. Moreover, $A_3^*B_3^*C_3^*$ is self-polar with respect to the polar conic of U.

8. **Osculating circle at U**

   Since we know the polar conic $\mathcal{C}_U$ of U, we are able to draw the osculating circle Γ_U at U to the cubic. Indeed, the curvature at U to the cubic is twice that at U to $\mathcal{C}_U$ (theorem of Moutard). Hence, if ρ_U is the center of the osculating circle at U to $\mathcal{C}_U$ then the center of Γ_U is R_U, midpoint of U, ρ_U. [10]

1.4.4 Construction of the asymptotes of a $p\mathcal{K}$

This construction is generally not possible with ruler and compass only : it needs to intersect conics and we must use Cabri-géomètre or equivalent.

First draw the polar conic of the pivot P (see remark 1 above) whose center is ω and the homothetic of this conic through $h_{P,1/2}$ intersecting the Ω-isoconjugate of $\mathcal{L}^\infty$ (see §1.2.2 final remark) at ω and three other points – one at least being real – denoted by E_i $(i = 1, 2, 3)$.
These three points lie on the cubic and are the isoconjugates of its points at infinity.
Then draw the reflections F_i of P about E_i (the points F_i lie on the polar conic $\mathcal{C}_P$ of P) and their isoconjugates F_i^*.
The asymptotes are the parallels at F_i^* to the lines PE_i.

1.5 Nonpivotal isocubics or $n\mathcal{K}$

1.5.1 Definitions and known properties

- An isocubic of the form $(n\mathcal{K})$

$$ux\,(ry^2 + qz^2) + vy\,(pz^2 + rx^2) + wz\,(qx^2 + py^2) + k\,xyz = 0 \qquad (1.3)$$

  is said to be **nonpivotal**. We shall call $P(u : v : w)$ the **root** and the real number k the **parameter** of the isocubic $n\mathcal{K}$. If the parameter is zero, we shall write $n\mathcal{K}_0$. Note that the root P is not necessarily on the cubic.

[10] Construction of ρ_U. Denote by:
– T the second intersection of the normal at U to $\mathcal{C}_U$ with $\mathcal{C}_U$,
– F the Fregier point of U with respect to $\mathcal{C}_U$,
– M the midpoint of UT,
then ρ_U is the harmonic conjugate of M with respect to U and F (after Roger Cuppens).

- The equation (1.3) rewrites as

$$\left(\frac{x}{u}+\frac{y}{v}+\frac{z}{w}\right)\left(\frac{p}{ux}+\frac{q}{vy}+\frac{r}{wz}\right)=\frac{p}{u^2}+\frac{q}{v^2}+\frac{r}{w^2}-\frac{k}{uvw} \tag{1.4}$$

  in which we recognize the equations of $\mathbb{P}(P)$ and the circumconic which is its isoconjugate $\mathbb{P}(\mathrm{P})^*$ i.e.

$$\frac{x}{u}+\frac{y}{v}+\frac{z}{w}=0$$

  and

$$\frac{p}{ux}+\frac{q}{vy}+\frac{r}{wz}=0$$

  respectively.

- When the root $P(u:v:w)$ is given, the equation (1.3) above defines a pencil of isocubics with pole $(p:q:r)$, all passing through A, B, C, the feet U, V, W of $\mathbb{P}(P)$, and all having tangents at A, B, C passing through the feet U', V', W' of $\mathbb{P}(P^*)$ where $P^* = \Omega$-isoconjugate of P. [11]

  From this we see that the root of a $n\mathcal{K}$ is in fact the trilinear pole of the line through the "third" intersections of the curve with the sidelines of ABC.

  Finally, note that the points U, V, W, U', V', W' and the tangents at A, B, C are independent of k.

- In the definition given above, the parameter k has no geometrical signification. It is better to define a $n\mathcal{K}$ with its root $P(u:v:w)$ and two isoconjugate points $Q(x_0:y_0:z_0)$ and $Q^*(x_1:y_1:z_1)$ [12] on the curve. In this case, the equation rewrites under the more symmetrical form :

$$\sum_{\text{cyclic}} u\,x\,(y_0\,z-z_0\,y)(y_1\,z-z_1\,y)=0$$

  This cubic becomes a $n\mathcal{K}_0$ if and only if P lies on the trilinear polar of the cevian product of Q and Q^*. These cubics are called "apolar cubics" : any two vertices of triangle ABC are conjugated with respect to the polar conic of the remaining vertex.

- Two isoconjugate points M and M^* on a $n\mathcal{K}$ share the same tangential denoted by $\widetilde{M}$ which is the isoconjugate of the third intersection of the line MM^* with the cubic.

  In particular, A and U share the tangential $\widetilde{A}$. $\widetilde{B}$ and $\widetilde{C}$ are defined likewise and these three points are collinear since they are the tangentials of three collinear points U, V, W. The line through $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ passes through the intersection of $\mathbb{P}(P)$ and $\mathbb{P}(P^*)$.

- The tangents at A, B, C are the sidelines of the anticevian triangle of P^* and the $n\mathcal{K}$ is tritangent at A, B, C to the circum-conic with perspector P^*, the conic which is the isoconjugate of $\mathbb{P}(P)$.

[11] In other words, the tangents at A, B, C are the sidelines of the anticevian triangle of P^*.
[12] These points define the isoconjugation.

1.5.2 Characterization of $n\mathcal{K}_0$ isocubics

Theorem : Any $n\mathcal{K}_0$ with pole Ω, root P can be considered as the locus of point M such that the points M and M^* are conjugated with respect to the circum-conic $\mathcal{C}_P$ with perspector P.

In fact, more generally, $\mathcal{C}_P$ and the diagonal conic $\mathcal{D}_\Omega$ which passes through the fixed points of the isoconjugation, G and the vertices of the antimedial triangle generate a pencil of conics [13]. The $n\mathcal{K}_0$ can also be considered as the locus of point M such that the points M and M^* are conjugated with respect to any conic (except $\mathcal{D}_\Omega$) of this pencil.
Note that all the conics of the pencil are rectangular hyperbolas if and only if Ω lies on the line GK and P on the orthic axis. Otherwise, the pencil contains only one rectangular hyperbola.
It contains a circle if and only if P lies on the parallel at K to $\mathbb{P}(\mathbf{t}\Omega)$. In this case, all the conics of the pencil have the same directions of axes.

We can add the following propositions :

- A $n\mathcal{K}$ with root P is an $n\mathcal{K}_0$ if and only if the tangents at the intercepts U, V, W of $\mathbb{P}(P)$ concur. If this is the case, the point of concurrence is P^*.
- When the cubic is defined with two isoconjugate points $Q(x_0 : y_0 : z_0)$ and $Q^*(x_1 : y_1 : z_1)$ as seen above, the condition for which it is a $n\mathcal{K}_0$ is that its root P lies on the trilinear polar of the the cevian product (see §1.2.2) $Q \star Q^*$ of Q and Q^*.
- Any $n\mathcal{K}_0$ can be considered as the locus of point M such that the pole of the line MM^* in the conic $ABCMM^*$ (which is its isoconjugate) lies on the trilinear polar of the isoconjugate of its root. This can be compared to end of §1.4.2. See several examples in §4.3.3 and §7.2.1.

1.5.3 Characterization of $n\mathcal{K}$ isocubics

Theorem : Any $n\mathcal{K}$ can be considered as the locus of point M such that the points M and M^* are conjugated with respect to at least one fixed circle.

If the cubic is defined by its root P and two isoconjugate points Q and Q^*, the circle is centered on the radical axis of the circles with diameters AU, BV, CW. Two different situations then can occur depending of the positions of Q and Q^*.

When the circle with diameter QQ^* does not belong to the pencil generated by the three previous circles, the required fixed circle has its center at the radical center of the circles with diameters AU, BV, CW, QQ^* and is orthogonal to these circles.
In other words, the Jacobian of the circle with diameter QQ^* and any two of the three circles above degenerates into the line at infinity and the fixed circle.
Conversely, if the circle is given, U is the intersection of the sideline BC and the polar line of A in this circle. V and W being defined similarly, the root P of the $n\mathcal{K}$ is the trilinear pole of the line passing through the three collinear points U, V, W. In

[13] With $\Omega = p : q : r$, this diagonal conic $\mathcal{D}_\Omega$ has equation

$$\sum_{\text{cyclic}} (q-r)x^2 = 0.$$

It has the same infinite points as the circumconic whose perspector is the infinite point of $\mathbb{P}(\mathbf{t}\Omega)$. Both conics are tangent at G to the line $G\Omega$.

order to obtain a $n\mathcal{K}$ passing through the given point Q, we must take its isoconjugate Q^* on the polar line of Q in the circle.

Now, if the pole Ω of the isoconjugation and the center of the circle are given, all the corresponding $n\mathcal{K}$ (when the radius varies) form a pencil of cubics which contains one $n\mathcal{K}_0$ and the degenerated cubic formed by $\mathcal{L}^\infty$ and the circum-conic with perspector Ω. This yields that all these cubics have the same points at infinity and the same common points with the circum-conic above.

One of the most interesting examples is obtained when the pole is K and when the circles are centered at O. The pencil is formed by equilateral isogonal cubics with asymptotes parallel to the sidelines of the Morley triangle and meeting the circumcircle at the vertices of the circumtangential triangle. These asymptotes form an equilateral triangle with center G. The root of any such cubic lies on the line GK. This pencil contains **K024** (a $n\mathcal{K}_0$), **K085** (a nodal cubic), **K098**, **K105** and the cubic that decomposes into the circumcircle and $\mathcal{L}^\infty$.

When the circle with diameter QQ^* belongs to the pencil generated by the three circles, the radical center above is not defined. In this case, the pole must lie on the orthic axis and one can find a pencil of fixed circles. This pencil is formed by all the circles orthogonal to those with diameters AU, BV, CW.

Any such cubic contains the infinite point of the Newton line passing through the midpoints of AU, BV, CW and the two common points of the circles with diameters AU, BV, CW. The cubic has two other perpendicular asymptotes intersecting on the radical axis of the latter circles.

When P and Q are given and when the pole Ω traverses the orthic axis, the corresponding cubics form a pencil and the two perpendicular asymptotes envelope the inscribed parabola with directrix the radical axis of the four circles, with perspector and focus the isotomic and isogonal conjugates of the infinite point of this radical axis.

A remarkable special case occurs when the root is the Lemoine point K since all the cubics are $n\mathcal{K}_0$. In such case, the pencil of fixed circles is that generated by the circumcircle, the nine point circle, the orthoptic circle of the Steiner inscribed ellipse, the polar circle, the orthocentroidal circle, etc, with radical axis the orthic axis. All these cubics belong to a same pencil and pass through A, B, C, the centers Ω_a, Ω_b, Ω_c of the Apollonius circles, X_{523} and two points P_1, P_2 on the Euler line which are inverses with respect to any circle of the pencil. P_1, P_2 are actually the antiorthocorrespondents of K.

One of the most interesting cubic is that with pole the barycentric product of X_{468} and X_{523} since it is the central cubic **K608**. See figure 1.1.

1.5.4 Construction of a $n\mathcal{K}$

Let P be the root and P^* its isoconjugate (these two points defining the isoconjugation).

Remember that $\mathbb{P}(P)$ and $\mathbb{P}(P^*)$ meet the sidelines of ABC at U, V, W (these three points on the cubic) and U', V', W' respectively.

We need an extra point Q on the cubic i.e. distinct from A, B, C, U, V, W. When the cubic is a $n\mathcal{K}_0$, it is convenient to take $\widetilde{A} = AU' \cap P^*U$ (or equivalently $\widetilde{B} = BV' \cap P^*V$ or $\widetilde{C} = CW' \cap P^*W$).

The cevians of Q meet the line UVW at Q_a, Q_b, Q_c. For any point m on UVW, let us denote by m' its homologue under the involution which swaps U and Q_a, V and Q_b, W and Q_c. The line Qm' meets the conic $ABCQ^*m^*$ (isoconjugate of the line Qm) at two points M, N on the $n\mathcal{K}$. Similarly, the line Qm meets the conic $ABCQ^*m'^*$ at the

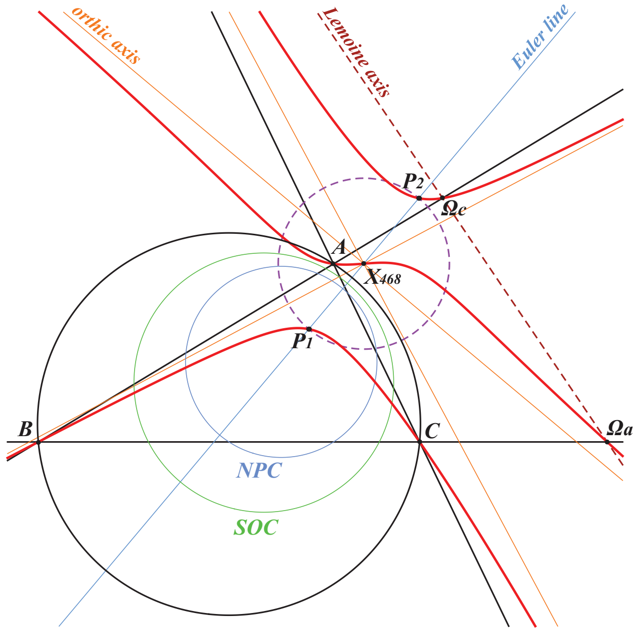

Figure 1.1: The central cubic **K608**

isoconjugates M^*, N^* of M, N and we have the following collinearities on the cubic :

$$Q, M, N; \quad Q, M^*, N^*; \quad Q^*, M, N^*; \quad Q^*, M^*, N.$$

In particular, we find the following points on the cubic :

$$A_1 = QA \cap Q^*U; \quad B_1 = QB \cap Q^*V; \quad C_1 = QC \cap Q^*W$$

and their isoconjugates :

$$A_1^* = QU \cap Q^*A; \quad B_1^* = QV \cap Q^*B; \quad C_1^* = QW \cap Q^*C.$$

Let $Z = MM^* \cap NN^*$. When m traverses the line UVW, the locus of Z is a line $\mathcal{L}_Z$ passing through the points $AU \cap A_1A_1^*$, $BV \cap B_1B_1^*$, $CW \cap C_1C_1^*$ and meeting the cubic at three points Z_1 (always real, on the line QQ^*) and two isoconjugate points Q_1, Q_2 which therefore lie on the conic isoconjugate of $\mathcal{L}_Z$. Note that Z_1^* is the tangential of Q and Q^* in the cubic and that the conic through Q, Q^*, Q_1, Q_2, Z_1^* is the polar conic of Z_1^* in the cubic.

The involution above has two fixed points F_1, F_2 on the line UVW which are not necessarily real. If they are, it is possible to draw the four tangents passing through Q^* to the cubic. The four contacts are the intersections of the lines QF_1, QF_2 with the conics $ABCQ^*F_1^*$, $ABCQ^*F_2^*$ respectively. Obviously, the diagonal triangle of the quadrilateral formed by these four points is the triangle QQ_1Q_2. This shows that the $n\mathcal{K}$ is in fact a $p\mathcal{K}$ with respect to this triangle and its pivot is Q^*. It is invariant in the isoconjugation (with respect to QQ_1Q_2) whose fixed points are the four contacts. This isoconjugation also swaps the points M, N^* and N, M^*. Z_1^* is naturally the secondary pivot of the $p\mathcal{K}$. Hence, it is possible to draw the $n\mathcal{K}$ with the construction seen in §1.4.3.

This construction can be generalized for any non-singular cubic as far as we are able to draw the four contacts of the tangents passing through any point on the cubic which is rarely possible in the most general case.

In more specific situations, other simpler constructions will be seen.

1.5.5 Isogonal $n\mathcal{K}$

Theorem : Any isogonal $n\mathcal{K}$ is, in a single way, the locus of point M whose pedal circle[14] is orthogonal to a fixed circle Γ whose center is denoted ω.

ω is the midpoint of H and the center of the circle defined in §1.5.3.

Remarks :

1. Γ can be real or imaginary and even reduced to a single point.
2. This isogonal $n\mathcal{K}$ is a $n\mathcal{K}_{60}$ if and only if $\omega = X_5$ (see Chapter 7).
3. This circle Γ can degenerate into the union of $\mathcal{L}^\infty$ and another line $\mathcal{L}$: in this case, we obtain an isogonal circular focal $n\mathcal{K}$, locus of M whose pedal circle is centered on $\mathcal{L}$ (see Chapter 4). This $n\mathcal{K}$ becomes a $n\mathcal{K}_0$ when $\mathcal{L}$ passes through K.
4. Let ℓ_H be a line through H meeting the Kiepert hyperbola again at H_2 and let ℓ_G be the line through G and H_2. For any ω on ℓ_H, the isogonal $n\mathcal{K}$ as seen in theorem above has its root P on the line ℓ_G.
   More precisely, for a given center ω, the root P lies on the line through G and the orthocorrespondent $\omega^\perp$ of ω. The $n\mathcal{K}_0$ is obtained when $P = \omega^\perp$.
   For example :
   - when P lies on the Euler line, ω lies on the tangent at H to the Kiepert hyperbola.
   - when P lies on the line GK, ω lies on the Euler line.
5. Any isogonal $n\mathcal{K}$ meets the circumcircle at A, B, C and three other points (one at least is real) whose Simson lines pass through the point ω.

1.5.6 Construction of an isogonal $n\mathcal{K}$ knowing the circle Γ

Γ is a circle centered at ω.

- The circle through A, the foot of the altitude H_a and the inversive image ${}^iA$ of A with respect to Γ is orthogonal to Γ and meets the line BC at H_a and another point U. Its center is denoted by ω_A. Similarly, the points V on CA and W on AB are defined and the three points are on a line whose trilinear pole is the root P of the $n\mathcal{K}$.

  As the radius of Γ varies, the point P traverses a line through G and the second meet of the line $H\omega$ with the Kiepert hyperbola.

- In order to make use of the construction seen in §1.5.4, we need an extra point on the cubic and we seek such a point Q on the line AU.

  Let E_A be the intersection of the altitude AH with the Simson line of the second intersection of AU' and the circumcircle. The midpoint of $Q\mathbf{g}Q$ lies on the perpendicular ℓ_A to ωE_A at ω_A. The hyperbola through ω_A, the midpoint of AU', the midpoint of $\omega_A\mathbf{g}\omega_A$ and whose asymptotes are parallel to the lines AU and AU' meets ℓ_A again at the midpoint q of $Q\mathbf{g}Q$. Now the circle centered at q orthogonal to Γ is the pedal circle of Q and $\mathbf{g}Q$. In order to get $\mathbf{g}Q$, reflect the line AU about q intersecting AU' at $\mathbf{g}Q$ and finally reflect $\mathbf{g}Q$ about q to get Q.

[14] Remember that two isogonal conjugates share the same pedal circle.

1.5.7 $n\mathcal{K}_0$ and nets of conics

Let $\Omega = p : q : r$ and $P = u : v : w$ be the pole and the root of a $n\mathcal{K}_0$.

Remember that :

– R_o, R_a, R_b, R_c are the square roots of Ω (see §1.2.2) i.e. the fixed points of the isoconjugation,

– U, V, W are the traces of $\mathbb{P}(P)$,

– U', V', W' are the traces of $\mathbb{P}(P^*)$.

Theorem : assuming that $pu^2+qv^2+rw^2 \neq 0$, any $n\mathcal{K}_0(\Omega, P)$ is the jacobian[15] of a net $\mathcal{N}$ of conics containing the circum-conic $\mathcal{C}_P$ with perspector P.

This net is generated by the three independent[16] conics γ_A, γ_B, γ_C with respective equations :

$$\gamma_A : \quad 2u(uyz + vzx + wxy) - vw(ry^2 - qz^2) = 0,$$

$$\gamma_B : \quad 2v(uyz + vzx + wxy) - wu(pz^2 - rx^2) = 0,$$

$$\gamma_C : \quad 2w(uyz + vzx + wxy) - uv(qx^2 - py^2) = 0,$$

where $uyz + vzx + wxy = 0$ is the equation of $\mathcal{C}_P$.

Thus, for any point M on the cubic, the polar lines of M in γ_A, γ_B, γ_C concur at M^* on the cubic.

γ_A passes through A and is tangent at this point to the corresponding sideline of the anticevian triangle of P (and therefore it is tangent at A to $\mathcal{C}_P$). It meets $\mathcal{C}_P$ again at two points lying on the two lines through A which contain the fixed points of the isoconjugation. These two points are in fact the intersections of $\mathcal{C}_P$ and the polar line of U' in $\mathcal{C}_P$. γ_A meets the lines AB, AC again at two points on the line δ_A with equation $2ux - rvy + qwz = 0$. This line contains the $A-$vertex of the cevian triangle of P^*.

This net $\mathcal{N}$ contains :

– only one circum-conic which is $\mathcal{C}_P$,

– only one (possibly degenerate) circle,

– a pencil of diagonal conics passing through R_o, R_a, R_b, R_c,

– a pencil of rectangular hyperbolas.

1.5.8 $n\mathcal{K}$ and nets of conics

Let $n\mathcal{K}(\Omega, P, Q)$ be the cubic with pole $\Omega = p : q : r$ and root $P = u : v : w$ passing through a given point $Q = \alpha : \beta : \gamma$ not lying on a sideline of ABC.

There is a net of conics such that this cubic is the locus of point M such that M and its $\Omega-$isoconjugate M^* are conjugated with respect to any conic of the net.

If we suppose that this cubic is not a $n\mathcal{K}_0$, the net is generated by the three independent conics :

$$\gamma_A : -2p\alpha\beta\gamma(uyz + vzx + wxy) + Tx^2 = 0,$$

$$\gamma_B : -2q\alpha\beta\gamma(uyz + vzx + wxy) + Ty^2 = 0,$$

$$\gamma_C : -2r\alpha\beta\gamma(uyz + vzx + wxy) + Tz^2 = 0,$$

where $T = \sum_{\text{cyclic}} p\beta\gamma(w\beta + v\gamma)$.

[15] The jacobian of a net of conics is the locus of points whose polar lines in the conics are concurrent (on the jacobian).

[16] When $pu^2 + qv^2 + rw^2 = 0$, these three conics are not independent i.e. any of them belongs to the pencil generated by the two others.

Note that $T = 0$ if and only if Q lies on $n\mathcal{K}_0(\Omega, P)$ which is here excluded.

Any conic of this net may be written under the form

$$T(\lambda x^2 + \mu y^2 + \nu z^2) - 2\alpha\beta\gamma(\lambda p + \mu q + \nu r)(uyz + vzx + wxy) = 0,$$

where λ, μ, ν are any three (not all zero) real numbers.

In this case, the cubic $n\mathcal{K}(\Omega, P, Q)$ is the jacobian of the net i.e. the jacobian of the conics γ_A, γ_B, γ_C.

Chapter 2

Poles and polars in a cubic

We will denote by $\mathcal{K}$ the cubic curve with equation $F(x, y, z) = F(M) = 0$ where F is a third degree homogeneous polynomial in x, y, z and where $M = (x, y, z)$.

Its partial derivatives will be noted F'_x for $\frac{\partial F}{\partial x}$ and F''_{xy} for $\frac{\partial^2 F}{\partial x \partial y}$ when no confusion is possible.

If P is a point in the triangle plane, $F'_x(P)$ is F'_x evaluated at P then,

$$\mathcal{G}_F(P) = \begin{pmatrix} F'_x(P) \\ F'_y(P) \\ F'_z(P) \end{pmatrix} \quad \text{and} \quad \mathcal{H}_F(P) = \begin{pmatrix} F''_{x^2}(P) & F''_{xy}(P) & F''_{xz}(P) \\ F''_{yx}(P) & F''_{y^2}(P) & F''_{yz}(P) \\ F''_{zx}(P) & F''_{zy}(P) & F''_{z^2}(P) \end{pmatrix}$$

are the gradient and the hessian matrices of F evaluated at P.

$\widetilde{\mathcal{H}_F}(P)$ is the comatrix (matrix of cofactors) of F evaluated at P.

At last, $\widehat{P}$ is P transposed and $\overline{PQ}$ is the signed distance from P to Q .

2.1 Polar line of a point in a cubic

2.1.1 Definition

Let P be a point. A variable line through P meets $\mathcal{K}$ at three points M_1, M_2, M_3.

The point Q on this line is defined by

$$\frac{3}{\overline{PQ}} = \frac{1}{\overline{PM_1}} + \frac{1}{\overline{PM_2}} + \frac{1}{\overline{PM_3}} \tag{2.1}$$

which is equivalent to

$$\frac{\overline{QM_1}}{\overline{PM_1}} + \frac{\overline{QM_2}}{\overline{PM_2}} + \frac{\overline{QM_3}}{\overline{PM_3}} = 0 \tag{2.2}$$

and the locus of Q is a straight line [1] called the **polar line** of the point P in the cubic : it will be denoted $\mathcal{L}_P$.

Its equation is :

$$M\,\mathcal{G}_F(P) = 0 \iff P\,\mathcal{H}_F(M)\,\widehat{P} = 0.^{2} \tag{2.3}$$

[1] This theorem was first given by Cotes in *Harmonia Mensurarum*. See [54] §132.

[2] The equation (2.3) can also be written under the equivalent forms :

$$M\,\mathcal{H}_F(P)\,\widehat{P} = 0 \iff P\,\mathcal{H}_F(P)\,\widehat{M} = 0.$$

2.1.2 Diameter in a cubic

When P is the point at infinity of a given line $\mathcal{L}$, equality (2.2) becomes

$$\overline{QM_1} + \overline{QM_2} + \overline{QM_3} = 0 \tag{2.4}$$

and Q is the isobarycentre of the points M_1, M_2, M_3.

In this case, the polar line of P is called the **associated diameter** of $\mathcal{L}$ in $\mathcal{K}$.

2.2 Polar conic of a point in a cubic

2.2.1 Definition

Let us now "reverse" the configuration and define the point Q on the same line through P, M_1, M_2, M_3 by :

$$\frac{\overline{PM_1}}{\overline{QM_1}} + \frac{\overline{PM_2}}{\overline{QM_2}} + \frac{\overline{PM_3}}{\overline{QM_3}} = 0 \iff \frac{3}{\overline{QP}} = \frac{1}{\overline{QM_1}} + \frac{1}{\overline{QM_2}} + \frac{1}{\overline{QM_3}} \tag{2.5}$$

The locus of Q is a conic called the **polar conic** of the point P in the cubic.

Its equation is :

$$P\,\mathcal{G}_F(M) = 0 \iff P\,\mathcal{H}_F(M)\,\widehat{M} = 0.^{3} \tag{2.6}$$

It will be denoted by $\mathcal{C}_P$. The matrix of this conic is $\mathcal{H}_F(P)$. $\mathcal{C}_P$ passes through the six (real or not) contacts of tangents drawn through P to $\mathcal{K}$. See figure 2.1.

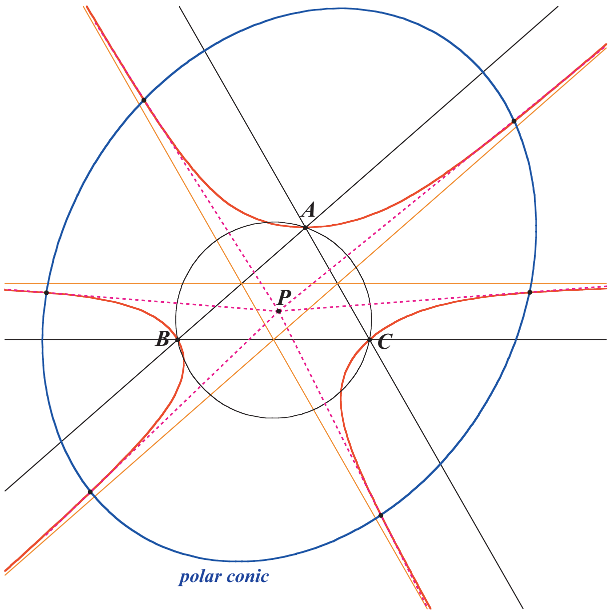

Figure 2.1: Polar conic with six real contacts

[3] The equation (2.6) can also be written under the equivalent forms :

$$M\,\mathcal{H}_F(P)\,\widehat{M} = 0 \iff M\,\mathcal{H}_F(M)\,\widehat{P} = 0.$$

Notice that $M\,\mathcal{H}_F(M)\,\widehat{M} = 0$ gives the cubic $\mathcal{K}$ itself.

2.2.2 Essential properties

- From equations (2.1) and (2.5), it is clear that Q lies on $\mathcal{C}_P$ if and only if P lies on $\mathcal{L}_Q$.
- The polar line of P in $\mathcal{C}_P$ is $\mathcal{L}_P$. See figure 2.2.

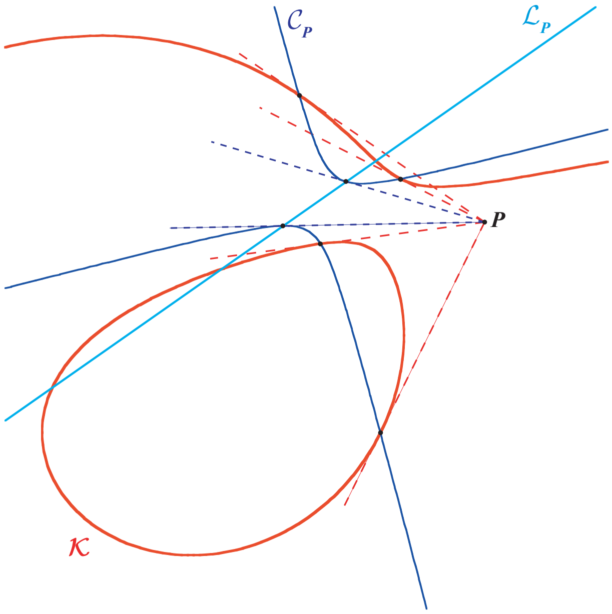

Figure 2.2: Polar line and polar conic

- When P is on the cubic, for instance $P = M_3$, equality (2.5) becomes :

$$\frac{\overline{PM_1}}{\overline{QM_1}} + \frac{\overline{PM_2}}{\overline{QM_2}} = 0 \tag{2.7}$$

  which is equivalent to $(P, Q, M_1, M_2) = -1$. Hence, $\mathcal{C}_P$ is the locus of Q, harmonic conjugate of P with respect to M_1 and M_2. In this case, $\mathcal{L}_P$ is the common tangent at P to $\mathcal{C}_P$ and $\mathcal{K}$.

- The equation (2.6) clearly shows that all the polar conics of the points of the plane form a net of conics generated by any three of them. It is convenient to take those of A, B, C namely $F'_x(M) = 0$, $F'_y(M) = 0$ and $F'_z(M) = 0$ when some computation is involved.

2.2.3 Diametral conic of a line in a cubic

When P is the point at infinity of a given line $\mathcal{L}$, equality (2.5) becomes

$$\frac{1}{\overline{QM_1}} + \frac{1}{\overline{QM_2}} + \frac{1}{\overline{QM_3}} = 0 \tag{2.8}$$

The locus of Q is a conic called **diametral conic** of $\mathcal{L}$ in $\mathcal{K}$.

2.2.4 Special polar conics

Let $\mathcal{C}_P$ be the polar conic of P in the cubic $\mathcal{K}$.

Circles

In general, there is one and only one point noted $\mathring{P}$ whose polar conic is a circle. Indeed, since the polar conic of $\mathring{P}$ contains the circular points at infinity J_1 and J_2, the polar lines of these two latter points are two imaginary conjugate lines of the form $\mathcal{L}_1 = D_1 + iD_2$, $\mathcal{L}_2 = D_1 - iD_2$ where D_1 and D_2 are usually two real lines. When D_1 and D_2 are defined (i.e. are proper lines) and distinct, $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect at a real point which is the required point $\mathring{P}$.

In the case of a circular cubic, $\mathring{P}$ is the singular focus of the cubic. See chapter 4.

If the cubic has three real asymptotes, $\mathring{P}$ is the Lemoine point of the triangle formed by the asymptotes and, obviously, when they concur, it is the point of concurrence. Furthermore, the parallels to these asymptotes passing through $\mathring{P}$ meet the cubic again at six points lying on a same circle. This circle is analogous to the (first) Lemoine circle obtained when the cubic is the union of the sidelines of triangle ABC.

Figure 2.3 shows the cubic **K414** (the Orthocubic of the excentral triangle) for which $\mathring{P} = X_9$. The polar conic of X_9 is the circle with center X_{649} passing through X_{15}, X_{16}, X_{1276}, X_{1277}. X_9 is the Lemoine point of the triangle formed by the asymptotes.

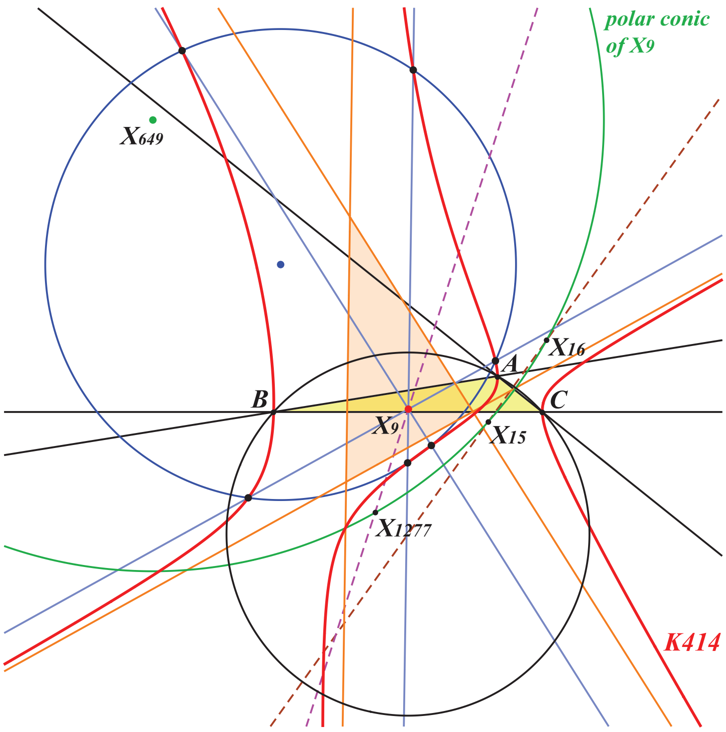

Figure 2.3: Circular polar conic

Now, if one can find three non collinear points whose polar conics are circles, the polar conics of all the points in the plane are circles and the cubic must decompose into the line at infinity and a circle. This happens when D_1 and D_2 both identically vanish.

When the polar conics of two distinct points are circles, all the points on the line $\mathcal{L}$ through these two points have also a circular polar conic. This line will be called the **circular line** of the cubic. Hence, the pencil of polar conics of the points of a line always contains a circle and the polar conics of all the points in the plane have the same directions of axes.

This occurs when
– (1) either one and only one of the lines D_1, D_2 identically vanishes,
– (2) or the lines D_1, D_2 coincide.

In both cases, the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ are real and coincide hence the cubic must have three concurring asymptotes but one only is real and the two other asymptotes are imaginary conjugates. The point of concurrence X lies on $\mathcal{L}$. The polar conic of X degenerates into the line at infinity and the radical axis of the pencil of the circular polar conics. See [58] for details and proofs.

In figure 2.4 is represented the cubic $n\mathcal{K}_0(X_{468}, X_{468})$ where the polar conics of all the points on the line GK are circles centered on the line GX_{98} and the polar conics of all the points not on GX_{98} have all the same directions of axes which are those of the asymptotes of the rectangular hyperbola with center X_{1560}, the orthojoin of X_{468}.

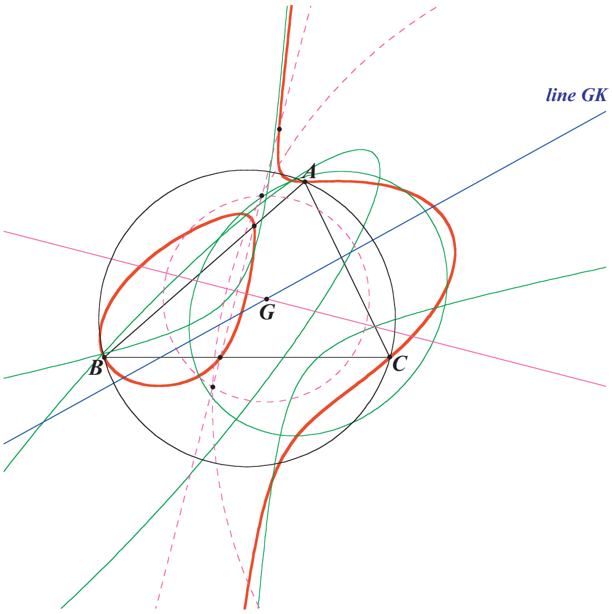

Figure 2.4: A cubic with a pencil of circular polar conics

More specific characterizations concerning $p\mathcal{K}$ and $n\mathcal{K}$ cubics are given in §2.4.1 and §2.4.2.

Rectangular hyperbolas

In general, the locus of P such as $\mathcal{C}_P$ is a rectangular hyperbola is a line sometimes called the orthic line of the cubic. This line is the polar line of $\mathring{P}$ in the poloconic of the line at infinity (see §2.3.4).

When the cubic has three real asymptotes, this poloconic is the Steiner inscribed ellipse in the triangle formed by these asymptotes and the orthic line of the cubic is the orthic axis of the triangle. Indeed, the polar line of the Lemoine point of a triangle in the Steiner inscribed ellipse of this same triangle is the orthic axis i.e. the trilinear polar of the orthocenter of the triangle.

But if one can find three non collinear points whose polar conics are rectangular hyperbolas then the polar conics of all the points in the plane are rectangular hyperbolas and the cubic is said to be an equilateral cubic denoted by $\mathcal{K}_{60}$ (see chapters 5, 6, 7).

Parabolas

$\mathcal{C}_P$ is a parabola if and only if P lies on the poloconic of the line at infinity (see §2.3.4).

2.2.5 Mixed polar line of two points in a cubic

Let P and Q be two points and $\mathcal{C}_P$ and $\mathcal{C}_Q$ their polar conics in the cubic $\mathcal{K}$. The polar lines of P in $\mathcal{C}_Q$ and Q in $\mathcal{C}_P$ coincide. This common polar line is called the mixed polar line of P and Q in $\mathcal{K}$ and is noted $\mathcal{L}_{P,Q}$.

The equation of $\mathcal{L}_{P,Q}$ is :

$$M\,\mathcal{G}_{F_P}(Q) = 0 \iff M\,\mathcal{G}_{F_Q}(P) = 0 \tag{2.9}$$

where $F_P = 0$ and $F_Q = 0$ are the equations of the polar lines of P in $\mathcal{C}_Q$ and Q in $\mathcal{C}_P$ respectively.

This equation can also be written under the forms :

$$P\,\mathcal{H}_F(M)\,\widehat{Q} = 0 \iff Q\,\mathcal{H}_F(M)\,\widehat{P} = 0 \tag{2.10}$$

2.2.6 Hessian of a cubic

The locus of point P such that $\mathcal{C}_P$ degenerates into two secant lines at Q is called the Hessian $\mathcal{H}$ of the cubic and then Q is also a point of the Hessian. P and Q are said to be two **corresponding points** (Salmon) or **conjugated points** (Durège) on the Hessian since the polar conic of Q also degenerates into two secant lines at P.
$\mathcal{H}$ is also the Jacobian of all the conics of the net we met above that is to say the locus of point P such that the polar lines of P with respect to any three of the conics of the net concur at Q. See figure 2.5 in which the three conics are $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and the polar lines of P are $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$.

The equation of the Hessian of $\mathcal{K}$ is :

$$det(\mathcal{H}_F(M)) = 0 \tag{2.11}$$

The tangents at P and Q to $\mathcal{H}$ meet at R, common tangential of P and Q. R lies on the Hessian and, obviously, the polar conic $\mathcal{C}_R$ of R in $\mathcal{K}$ degenerates into two lines secant at S (one of them is PQ, the other $P'Q'$ and P', Q' are also corresponding points on the Hessian) and then S is called **complementary point** of P and Q (and also P' and Q'). See figure 2.6.

We notice that the polar line of P in $\mathcal{K}$ is the tangent at Q to $\mathcal{H}$ and vice versa. Furthermore, the lines QP and QR are harmonic conjugates with respect to the two lines forming the polar conic of P. At last, R and S are also corresponding points on the Hessian.

It is a known fact (see [7, 54] for example) that $\mathcal{H}$ and $\mathcal{K}$ meet at the nine common inflexion points of the two cubics. Three are real and six are imaginary. The polar conic of each inflexion point I degenerates into two lines, one of them being the inflexional tangent at I and the other the **harmonic polar line** of I in the cubic.

2.2.7 Prehessians of a cubic

Any non-singular cubic $\mathcal{K}$ can be seen as the Hessian of three cubics $\Omega_i, i = 1, 2, 3$ (see [7, 54]). We shall say that Ω_i are the **prehessians** of $\mathcal{K}$ although these prehessians might not be all real.

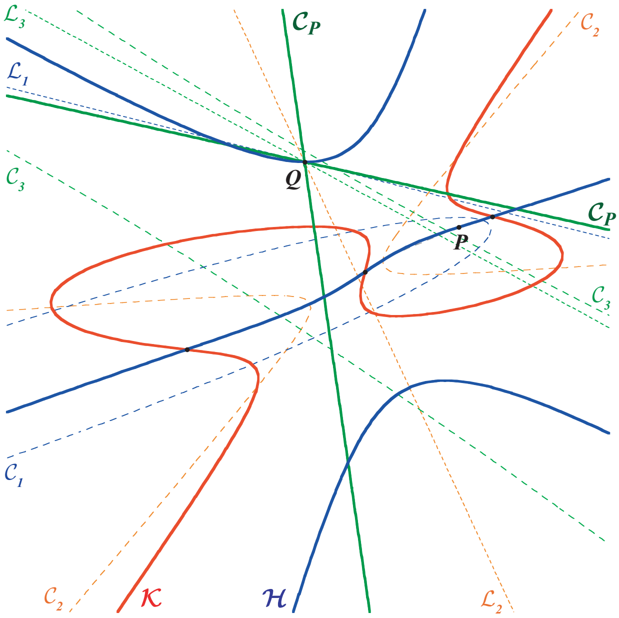

Figure 2.5: Hessian of a cubic

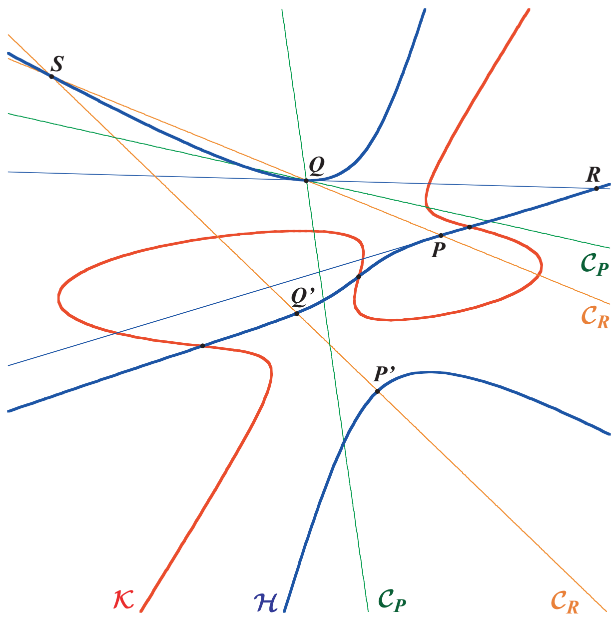

Figure 2.6: Corresponding and complementary points

Naturally, $\mathcal{K}$, its Hessian and its three prehessians belong to the same pencil of cubics since they pass through their nine common inflexion points. This pencil is called the syzygetic pencil of cubics associated with $\mathcal{K}$. See figure 2.7.

All that have been said in the paragraph above can be "reversed" and adapted to these three prehessians and we see that each point P on $\mathcal{K}$ has three corresponding points $Q = Q_1$, Q_2 and Q_3 according to the relative prehessians. We indeed can draw from R four tangents to $\mathcal{K}$, two of them being the lines RQ and RP, the remaining two RQ_2

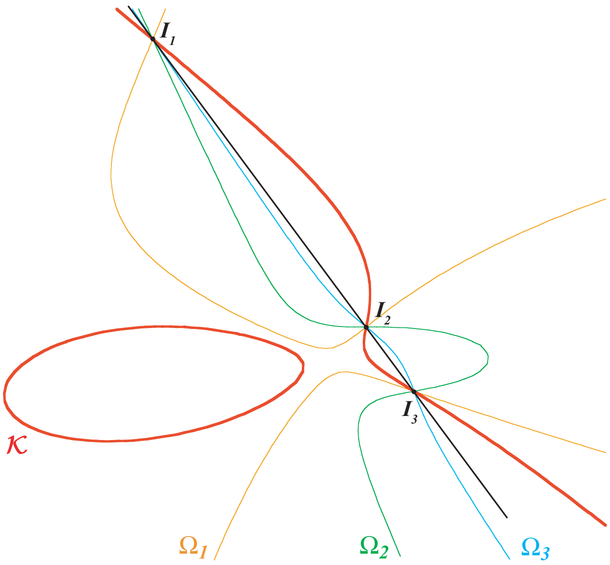

Figure 2.7: Prehessians of a cubic with three real collinear common inflexion points

and RQ_3 where Q_2 and Q_3 are the two points on $\mathcal{C}_R$ and $\mathcal{K}$ different of P, Q, S. See figure 2.8.

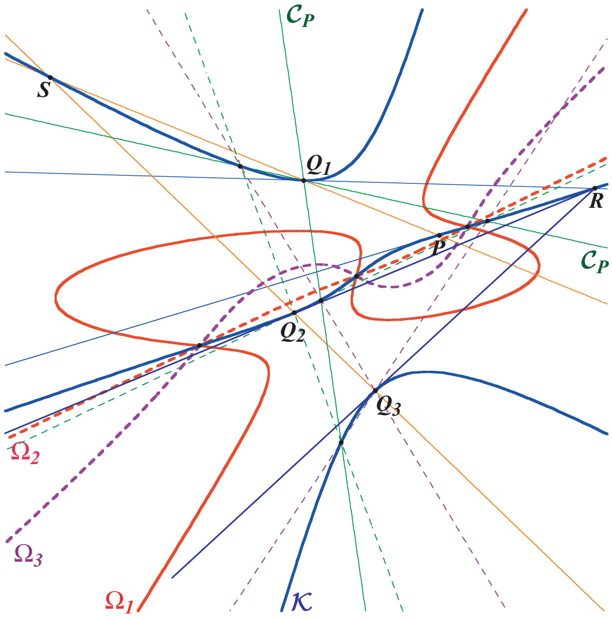

Figure 2.8: Corresponding points in the three prehessians of a cubic

In other words, Q_1, Q_2 and Q_3 are the "centers" of the three (degenerate) polar conics of P with respect to the three prehessians Ω_1, Ω_2, Ω_3 respectively. These three degenerate polar conics consist in six lines forming the complete quadrilateral $PQ_1Q_2Q_3$ whose diagonal triangle is formed by the three complementary points $S = S_1$, S_2 and S_3. P, Q_1, Q_2 and Q_3 are called the **pretangentials** of R. See figure 2.9.

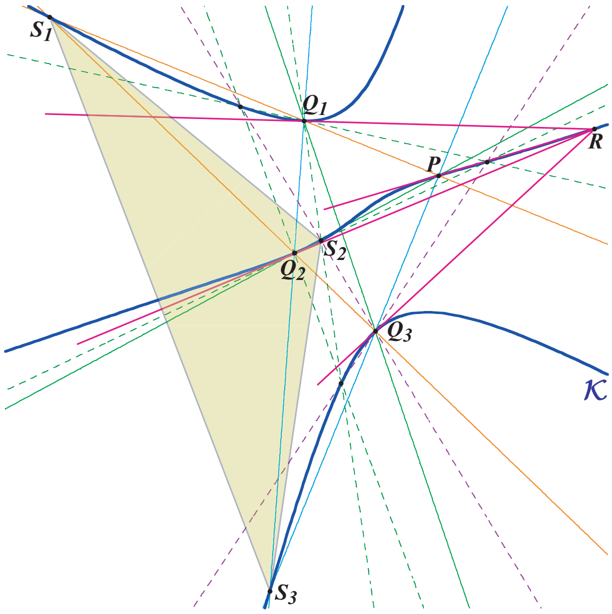

Figure 2.9: Complementary points and diagonal triangle

2.3 Poles and poloconic of a line in a cubic

2.3.1 Poles of a line in a cubic

Let $\mathcal{L}$ be a line. Each point M on $\mathcal{L}$ has a polar conic $\mathcal{C}_M$ in $\mathcal{K}$. The conics $\mathcal{C}_M$ form a pencil and therefore have four (not necessarily real) points Ω_o, Ω_a, Ω_b, Ω_b in common which are called the **poles** of the line $\mathcal{L}$ in $\mathcal{K}$.

These poles form a complete quadrilateral whose diagonal triangle $\mathcal{T} = T_aT_bT_c$ is self-polar in the pencil of conics. When the poles are all real, Ω_o is the one inside this diagonal triangle and the three other form a triangle which is the anticevian triangle of Ω_o with respect to $\mathcal{T}$. See figure 2.10. See [54] p.51 §61 and p.142 §165 for more informations.

2.3.2 Poloconic of a line in a cubic

- Let $\mathcal{L}$ be a line. Any point M on $\mathcal{L}$ has a polar line $\mathcal{L}_M$ in $\mathcal{K}$. The **poloconic** [4] of $\mathcal{L}$ in $\mathcal{K}$ is the envelope of $\mathcal{L}_M$ when M traverses $\mathcal{L}$. It will be denoted $\mathcal{C}_\mathcal{L}$. See [54] p. 184 & sq. for details and proofs.

  If $\mathcal{L}$ has equation $ux + vy + wz = 0$ where $L = (u, v, w)$ are the line coordinates, then an equation of $\mathcal{C}_\mathcal{L}$ is :
  $$L\,\widetilde{\mathcal{H}_F}(M)\,\widehat{L} = 0 \qquad (2.12)$$

- The poloconic of $\mathcal{L}$ is also :
  - the locus of the poles of $\mathcal{L}$ with respect to the polar conics of the points of $\mathcal{L}$.
  - the locus of the points whose polar conic is tangent to $\mathcal{L}$.

[4]This is called polar conic of a line in [54] and "poloconique" in [5]

2.3.3 Some propositions

Proposition 1

If Ω_o, Ω_a, Ω_b, Ω_b are the poles of the line $\mathcal{L}$ in a cubic $\mathcal{K}$, the poloconic $\mathcal{C}_\mathcal{L}$ passes through :

- the three vertices T_a, T_b, T_c of the diagonal triangle of the complete quadrilateral $\Omega_o\Omega_a\Omega_b\Omega_b$.
- the six harmonic conjugates of the intersections of $\mathcal{L}$ with a sideline of $\Omega_o\Omega_a\Omega_b\Omega_b$ with respect to the two vertices of $\Omega_o\Omega_a\Omega_b\Omega_b$ on the sideline.
- the two fixed points of the involution on $\mathcal{L}$ generated by the pencil $\mathcal{F}$ of conics through Ω_o, Ω_a, Ω_b, Ω_b. (these two points can be real or imaginary)

The poloconic of $\mathcal{L}$ is therefore the 11-point conic of $\mathcal{F}$. See figure 2.10.

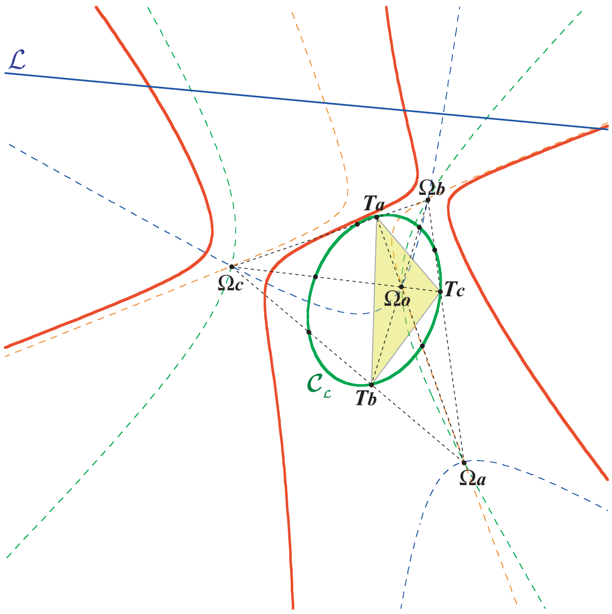

Figure 2.10: Poles and poloconic of a line in a cubic

Proposition 2

If $\mathcal{L}$ intersects $\mathcal{K}$ in three points X, Y, Z, the tangents t_X, t_Y, t_Z to $\mathcal{K}$ at these points are the tangents to the polar conics $\mathcal{C}_X$, $\mathcal{C}_Y$, $\mathcal{C}_Z$ at X, Y, Z. They form a triangle noted $X'Y'Z'$. Hence, the poloconic $\mathcal{C}_\mathcal{L}$ is inscribed in $X'Y'Z'$ and its perspector is the pole W of $\mathcal{L}$ in $\mathcal{C}_\mathcal{L}$. See figure 2.11.

Proposition 3

When the tangents t_X, t_Y, t_Z concur at a point W, the poloconic $\mathcal{C}_\mathcal{L}$ is degenerated into two straight lines through W. These lines are imaginary when X, Y, Z are real.

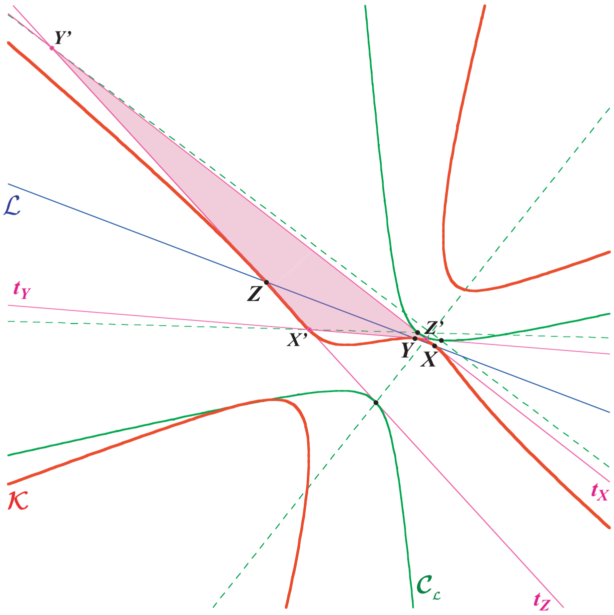

Figure 2.11: Proposition 2

Proposition 4

When $\mathcal{L}$ is tangent at X to $\mathcal{K}$ and meets $\mathcal{K}$ again at $\widetilde{X}$ (which is called the tangential of X), the poloconic of $\mathcal{L}$ is tangent to $\mathcal{L}$ at X and moreover tangent to the tangent to $\mathcal{K}$ at $\widetilde{X}$. See figure 2.12.

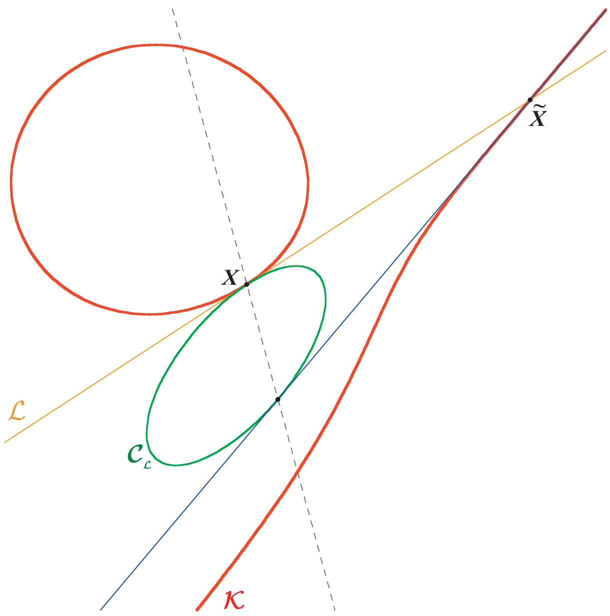

Figure 2.12: Proposition 4

Proposition 5

The poloconic $\mathcal{C}_\mathcal{L}$ of a line $\mathcal{L}$ is tritangent to the hessian $\mathcal{H}$ of the cubic $\mathcal{K}$. If $\mathcal{L}$ meets $\mathcal{H}$ at X, Y, Z then the points of tangency are the centers X', Y', Z' of the (degenerate) polar conics of X, Y, Z with respect to $\mathcal{K}$. These points X', Y', Z' lie on $\mathcal{H}$ and are said to be the conjugate points of X, Y, Z with respect to $\mathcal{H}$.

$X'Y'Z'$ is the diagonal triangle of the complete quadrilateral triangle formed by the four poles of $\mathcal{L}$ with respect to $\mathcal{K}$. See figure 2.13.

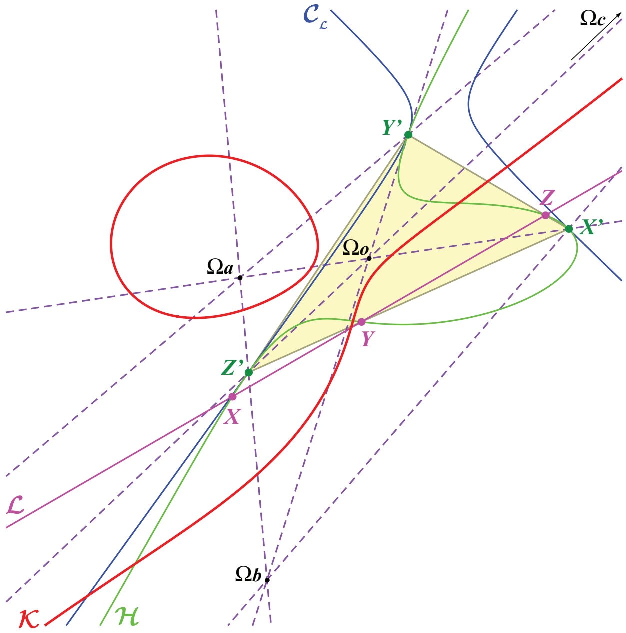

Figure 2.13: Proposition 5

2.3.4 Poloconic of $\mathcal{L}^\infty$

The general properties given above show that $\mathcal{C}_{\mathcal{L}^\infty}$, the poloconic of $\mathcal{L}^\infty$, is :

- the envelope of the diameters of the cubic i.e. the envelope of the polar lines of the points on $\mathcal{L}^\infty$.
- the locus of the centers of the diametral conics of the cubic i.e. the polar conics of the points on $\mathcal{L}^\infty$.
- the locus of the points whose polar conic is a parabola.

$\mathcal{C}_{\mathcal{L}^\infty}$ contains the six midpoints of the quadrilateral formed by the poles Ω_o, Ω_a, Ω_b, Ω_b of $\mathcal{L}^\infty$ and the vertices of the diagonal triangle. The center of $\mathcal{C}_{\mathcal{L}^\infty}$ is the isobarycenter of these four poles i.e. the common midpoint of the three segments joining the six midpoints above. See figure 2.14.

When the cubic has three real asymptotes, $\mathcal{C}_{\mathcal{L}^\infty}$ is the inscribed Steiner ellipse of the triangle formed with the three asymptotes (Take $\mathcal{L} = \mathcal{L}^\infty$ in proposition 2 above).

$\mathcal{C}_{\mathcal{L}^\infty}$ is degenerate if and only if the cubic is a $\mathcal{K}^+$ i.e. a cubic with concurring asymptotes.

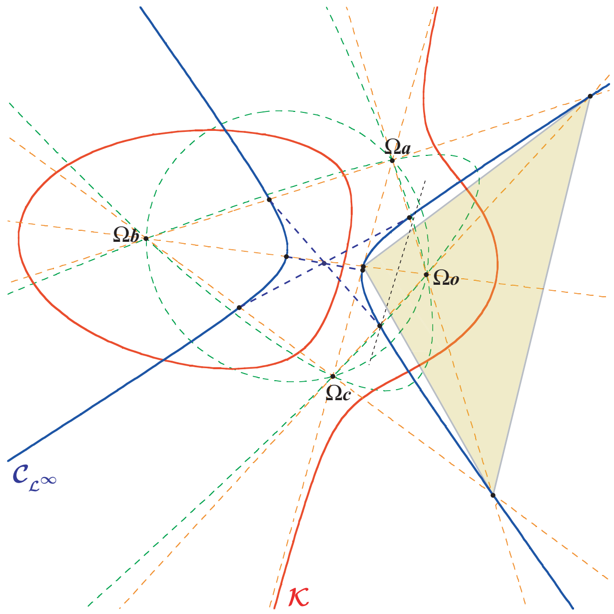

Figure 2.14: Poloconic of $\mathcal{L}^\infty$

When $\mathcal{C}_{\mathcal{L}^\infty}$ is a circle, this Steiner ellipse must be the incircle of the triangle formed with the three asymptotes hence the three asymptotes make 60° angles with one another and the cubic is a $\mathcal{K}_{60}$. See chapters 5, 6 ,7.

2.3.5 Mixed poloconic of two lines in a cubic

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lines with line coordinates L_1 and L_2. The locus of the poles of $\mathcal{L}_1$ with respect to the polar conics of the points of $\mathcal{L}_2$ and the locus of the poles of $\mathcal{L}_2$ with respect to the polar conics of the points of $\mathcal{L}_1$ coincide. This locus is a conic called **mixed poloconic** of the two lines in the cubic. Obviously, when $\mathcal{L}_1 = \mathcal{L}_2$, we find the "ordinary" poloconic of the line in the cubic. The equation is :

$$L_1\,\widetilde{\mathcal{H}_F}(M)\,\widehat{L_2} = 0 \iff L_2\,\widetilde{\mathcal{H}_F}(M)\,\widehat{L_1} = 0 \qquad (2.13)$$

2.4 Application to $p\mathcal{K}$ isocubics

Let us consider the pivotal isocubic $p\mathcal{K}$ with pole $\Omega = (p, q, r)$ and pivot $P = (u, v, w)$ i.e. the locus of point M such that M, its isoconjugate M^* and P are collinear.

2.4.1 Polar conics

Diagonal and circumscribed polar conics

If P is not one of the fixed points of the isoconjugation [5] that is $P^2 \neq \Omega$, then :

- there is one and only one diagonal polar conic and it is that of the pivot P.

  Its equation is :

$$\sum_{\text{cyclic}} (rv^2 - qw^2)\, x^2 = 0$$

[5] when $\Omega = P^2$ the cubic degenerates into the three cevians of P.

It passes through P, the vertices of the anticevian triangle of P and the four fixed points of the isoconjugation. It also contains the vertices of the anticevian triangle of any of its points. It is tangent at P to the line PP^*.

- there is one and only one circumscribed polar conic and it is that of the isoconjugate P^* of the pivot.

  Its equation is :

$$\sum_{\text{cyclic}} p\,u\,(rv^2 - qw^2)\,y\,z = 0$$

  It passes through A, B, C, P and P^*. The tangents at P and P^* pass through P^*/P and P/P^* respectively.

Rectangular hyperbolas

In general, the locus of point M whose polar conic is a rectangular hyperbola is the line – sometimes called the orthic line of the cubic – with equation :

$$\sum_{\text{cyclic}} \left(2S_C rv - 2S_B qw + b^2 ru - c^2 qu\right) x = 0$$

When the cubic has three real asymptotes, this line is the orthic axis of the triangle formed by these asymptotes.

Consequently, there are exactly three points (at least one is real) on the curve whose polar conic is a rectangular hyperbola.

But when the pivot P lies on the Neuberg cubic and the pole lies on a cubic we will call $\mathcal{C}_o$ or **K095**, every point in the plane has a polar conic which is a rectangular hyperbola.

This will lead us to $p\mathcal{K}_{60}$ isocubics or equilateral pivotal isocubics and this will be detailed in Chapter 6.

Circular polar conics

In general, there is one and only one point whose polar conic is a circle. When the cubic is circular, it is the singular focus. When it has three concurring asymptotes, it is the point of concurrence. When the cubic has three real asymptotes, this point is the Lemoine point of the triangle formed by these asymptotes.

If there are two distinct points whose polar conics are circles then the polar conic of any point on the line $\mathcal{L}$ passing through these two points must be a circle. Indeed, recall that the polar conics of the points lying on a same line form a pencil of conics passing through the poles of the line in the cubic. In this case, the pivotal cubic must have three (not necessarily real) concurring asymptotes and the point of concurrence must lie on $\mathcal{L}$.

This, in particular, occurs when the pole Ω of the cubic lies on the orthic axis and the pivot P on the nine point circle, being the center of the rectangular circum-hyperbola with perspector Ω. The asymptotes always concur at the centroid G of ABC.

The figure 2.15 shows **K392** $= p\mathcal{K}(X_{523}, X_{115})$ where X_{523} is the perspector of the Kiepert hyperbola and X_{115} its center. The polar conic of any point on the line (L) passing through G, X_{1637}, X_{2799} is a circle.

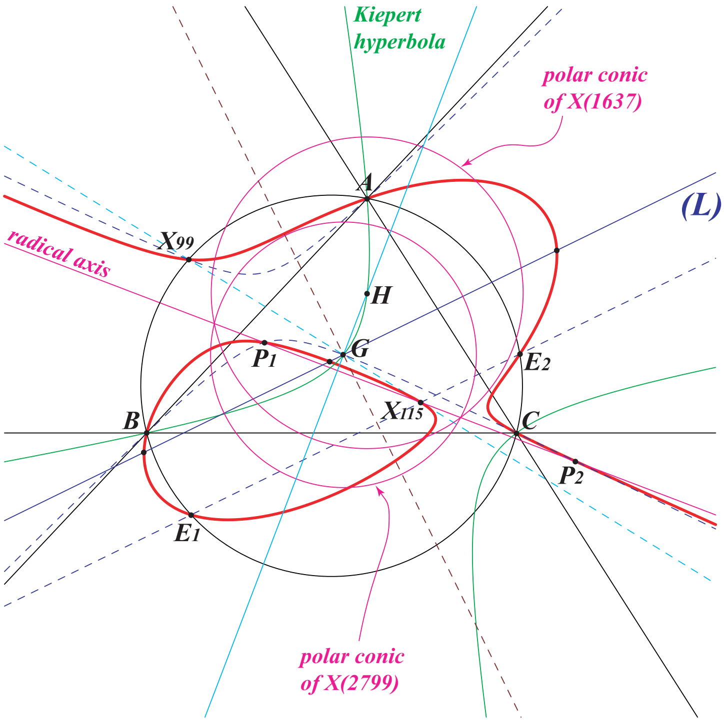

Figure 2.15: A pivotal cubic having a pencil of circular polar conics

2.4.2 Prehessians

Proposition : Every non degenerate $p\mathcal{K}$ has always three real prehessians.

Proof : since a prehessian $\mathcal{P}_i$ of $\mathcal{K}$ belongs to the pencil of cubics generated by the cubic $\mathcal{K}$ itself and its Hessian $\mathcal{H}$, we seek $\mathcal{P}_i$ under the form $(1-t)\,\mathcal{K}+t\,\mathcal{H}$ and write that the Hessian of $\mathcal{P}_i$ must be $\mathcal{K}$. This gives a third degree equation in t which always factorizes into three first degree factors.

The figure 2.16 shows the McCay cubic **K003**, its Hessian **K048** – a circular cubic – and its three prehessians $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$.

It follows that there are three conjugations $\mathcal{F}_i$ leaving the cubic $p\mathcal{K}$ invariant : $\mathcal{F}_i$ maps any point M onto the center of the polar conic of M with respect to $\mathcal{P}_i$.

Thus, for any point M on the cubic $p\mathcal{K}$, one can find three points M_1, M_2, M_3 also lying on the cubic such that $M_i = \mathcal{F}_i(M)$. The triangle $M_1M_2M_3$ is inscribed in the polar conic of M^* and autopolar in the polar conic of $\mathcal{C}_M$ of M. Indeed, $\mathcal{C}_M$ meets the cubic at M (counted twice) and four other points forming a complete quadrilateral whose diagonal triangle is $M_1M_2M_3$.

Furthermore, the tangents at M, M_1, M_2, M_3 concur at the tangential M' of M. Obviously, the polar conic of M' contains M, M_1, M_2, M_3, M'.

Consequently, the cubic is a pivotal cubic with pivot M, isopivot M' with respect to the triangle $M_1M_2M_3$. This can be realized in infinitely many ways as far as M is not a flex on the cubic. See figure 2.17 where the Neuberg cubic is represented with one of these triangles $M_1M_2M_3$.

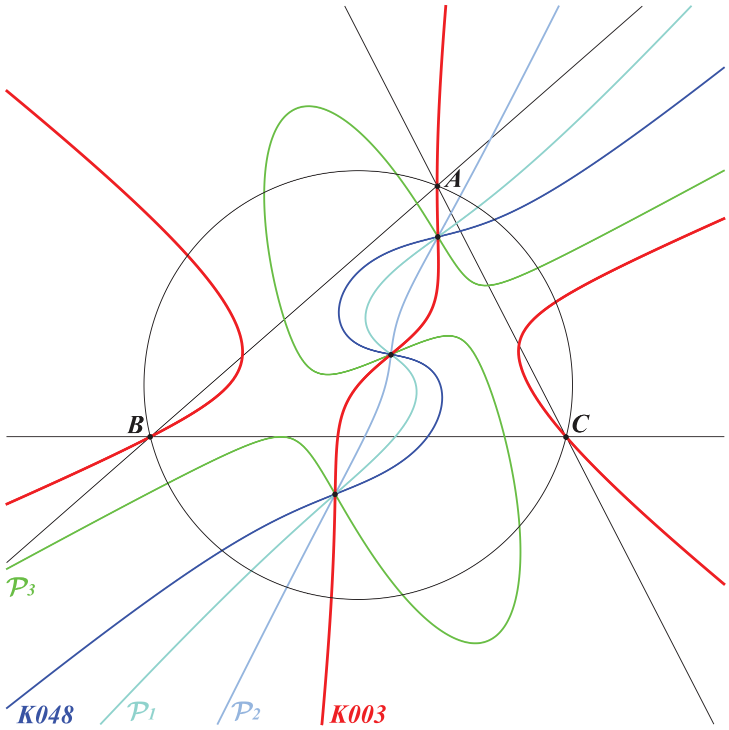

Figure 2.16: Hessian and prehessians

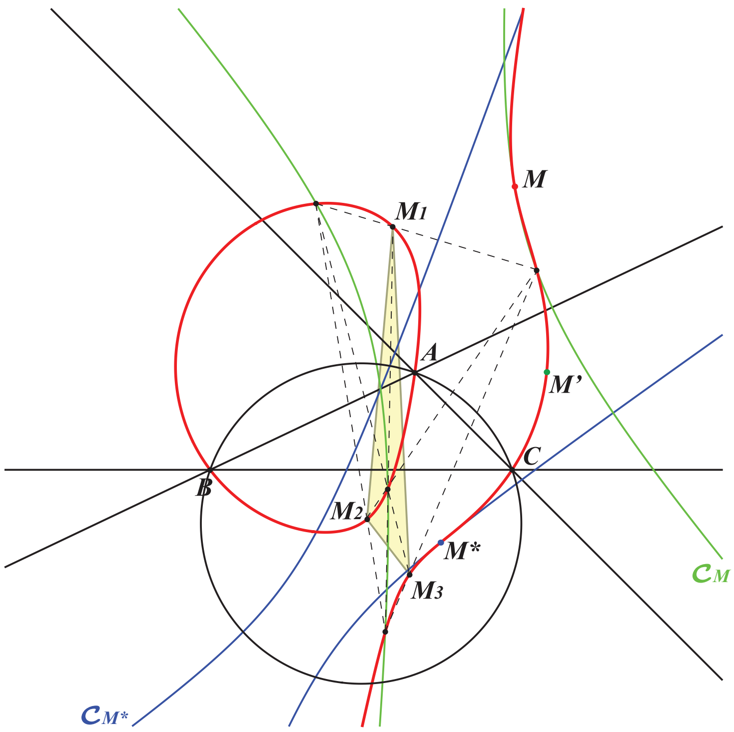

Figure 2.17: The Neuberg cubic with one triangle $M_1M_2M_3$

The Feuerbach theorem for pivotal cubics

This section is a consequence and an illustration of Malgouzou's paper. See [44].

Let Δ_i be the orthic line of the prehessian cubic $\mathcal{P}_i$. This line is unique as far as $\mathcal{P}_i$ is not a stelloid. See chapter 5.

The mixed polar conic of the line at infinity and Δ_i is a circle we shall call the Euler circle of $\mathcal{P}_i$. See §2.6.2 below. If $\mathcal{T}_i$ is the diagonal triangle of the four poles of Δ_i with respect to $\mathcal{P}_i$ then the isoconjugation with respect to $\mathcal{T}_i$ whose fixed points are these four poles maps the line at infinity to this Euler circle.

Recall that the poloconic of any line with respect to $\mathcal{P}_i$ is a conic tritangent to the pivotal cubic $p\mathcal{K}$. There are four lines (not necessarily real) for which this conic is a circle therefore there are four circles tritangent to the $p\mathcal{K}$ and also tangent to the Euler circle of $\mathcal{P}_i$. It follows that there are three groups of four circles (analogous to the in/excircles of ABC) tritangent to the $p\mathcal{K}$ and tangent to one of the three Euler circles of the prehessians.

Figure 2.18 shows a configuration with the Lucas cubic **K007** where the four tritangent circles are all real although all the contacts with the cubic are not necessarily real.

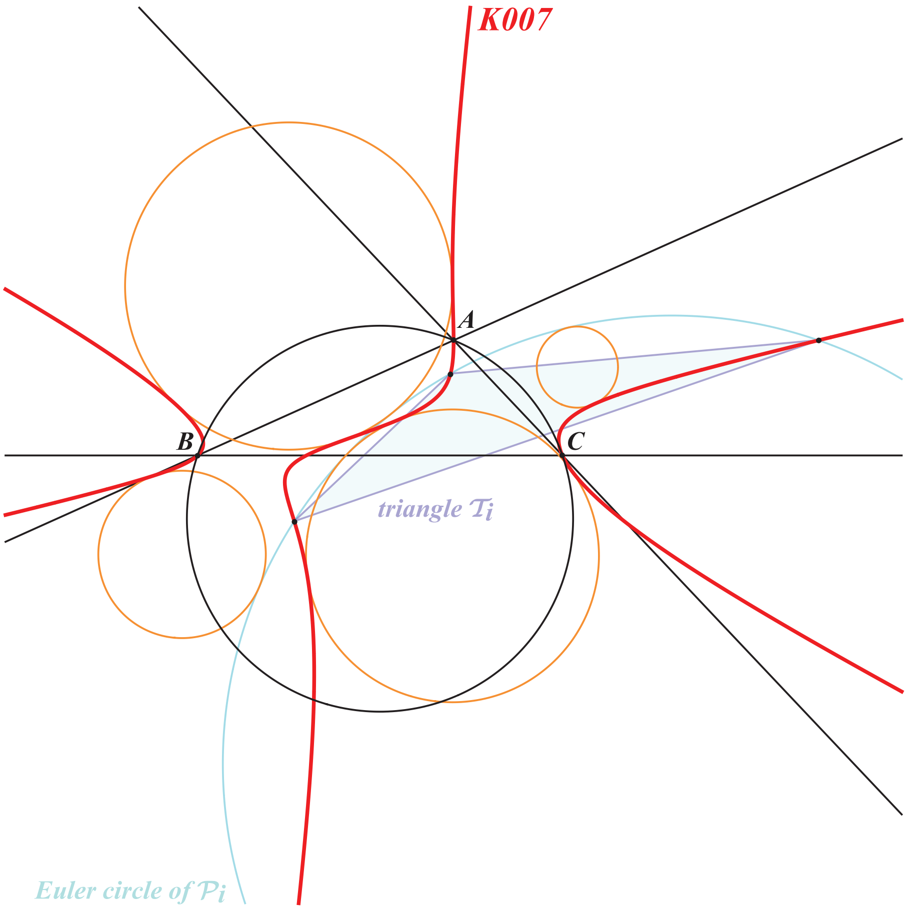

Figure 2.18: The Feuerbach theorem for the Lucas cubic

2.5 Application to $n\mathcal{K}_0$ isocubics

Let us consider a non-pivotal isocubic $n\mathcal{K}_0$ with pole $\Omega = (p, q, r)$ and root $P = (u, v, w)$. Recall that such a cubic always contains the feet U, V, W of the trilinear polar of P on the sidelines of ABC.

2.5.1 Polar conics

Diagonal and circumscribed polar conics

In general, one cannot find a point whose polar conic in a $n\mathcal{K}_0$ is a circum-conic or a diagonal conic.

This occurs when either the pole Ω or the root P lies on a sideline of ABC. In the former case, the cubic decomposes into this sideline and a conic through the remaining vertex of ABC. In the latter case, when P lies on BC for instance, the polar conic of A is a circum-conic but decomposed into BC and a line through A and the polar conic of U is diagonal.

Rectangular hyperbolas

The results obtained for pivotal isocubics are easily adapted. In general, the locus of point M whose polar conic is a rectangular hyperbola is the (orthic) line with equation :

$$\sum_{\text{cyclic}} \left(2S_C rv + 2S_B qw - b^2 ru - c^2 qu\right) x = 0$$

When the cubic has three real asymptotes, this line is the orthic axis of the triangle formed by these asymptotes.

Consequently, there are exactly three points (at least one is real) on the curve whose polar conic is a rectangular hyperbola.

But when the pole lies on the cubic **K396** $= n\mathcal{K}(X_6 \times X_{1989}, X_{1989}, X_6)$ and the root P lies on **K397** $= n\mathcal{K}(X_6, X_{30}, X_2)$, every point in the plane has a polar conic which is a rectangular hyperbola.

This will lead us to $n\mathcal{K}_{60}$ isocubics or equilateral non-pivotal isocubics and this will be detailed in Chapter 7.

Circular polar conics

Here again we obtain similar results to those for pivotal isocubics and, in general, there is one and only one point whose polar conic is a circle.

However, there are several remarkable families of $n\mathcal{K}_0$ cubics having a pencil of circular polar conics.

- Every $n\mathcal{K}_0(\Omega, \Omega)$ cubic with pole and root Ω on the orthic axis has a pencil of circular polar conics.

  This cubic has three (not all real) asymptotes concurring at G and the polar conic of G decomposes into $\mathcal{L}^\infty$ and the trilinear polar of Ω which is the radical axis of the pencil of circles.

  The orthic line also contains G and its infinite point is the $\Omega-$isoconjugate of H. One remarkable thing to observe is that the polar conic of this latter infinite point is the orthoptic circle of the Steiner inscribed ellipse for every point Ω on the orthic axis.

When $\Omega = X_{1990}$ the orthic line is the Euler line and when $\Omega = X_{468}$ the orthic line is the line GK.

The figure 2.19 presents $n\mathcal{K}_0(X_{523}, X_{523})$ where X_{523} is the infinite point of the orthic axis. The orthic line contains X_2, X_{525}, X_{1640}, X_{2394}, X_{2433}. The radical axis is the trilinear polar of X_{523} passing through the centers of the Kiepert and Jerabek hyperbolas namely X_{115} and X_{125} respectively.

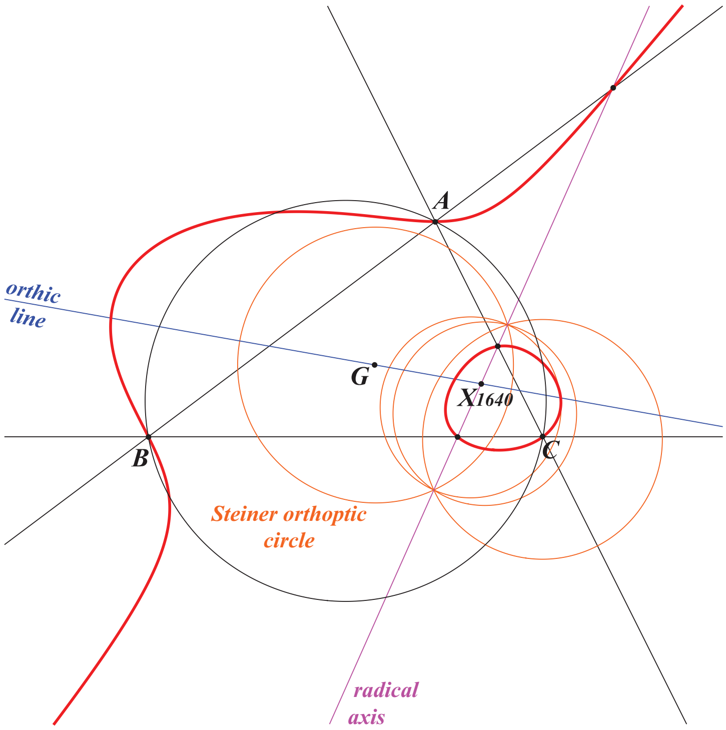

Figure 2.19: $n\mathcal{K}_0(X_{523}, X_{523})$, a non-pivotal cubic having a pencil of circular polar conics

- Every $n\mathcal{K}_0(\Omega, P)$ cubic with root $P = u : v : w \neq G$ on the Thomson cubic **K002** and corresponding pole $\Omega = u(v + w - 2u)$ :: has a pencil of circular polar conics.

  The asymptotes concur at the tripolar centroid $TC(P)$ of P [6] and the polar conic of this point decomposes into $\mathcal{L}^\infty$ and the radical axis of the pencil of circles. Naturally, the orthic line also contains $TC(P)$.

  The figure 2.20 shows **K393** $= n\mathcal{K}_0(X_{1990}, X_4)$. All the points on the orthic axis of ABC have a circular polar conic. The radical axis is the parallel at X_{1990} to the Euler line. X_{1990} lies on the line HK. It is the barycentric product of H and X_{30}, the infinite point of the Euler line.

  The orthic axis meets the sidelines at U, V, W and X_{1637} is $TC(H)$. The polar conic of X_{1637} is the degenerate circle into the line at infinity and the radical axis.

  The polar conic of X_{523}, the infinite point of the orthic axis, is the circle with center X_{1637} passing through the Fermat points and orthogonal to the circumcircle.

  For any finite point P on the orthic axis, distinct of X_{1637}, the polar conic is a proper circle whose center is the inverse of P in the circle above.

[6] The tripolar centroid of a point P is the isobarycenter of the traces of its trilinear polar on the sidelines of ABC.

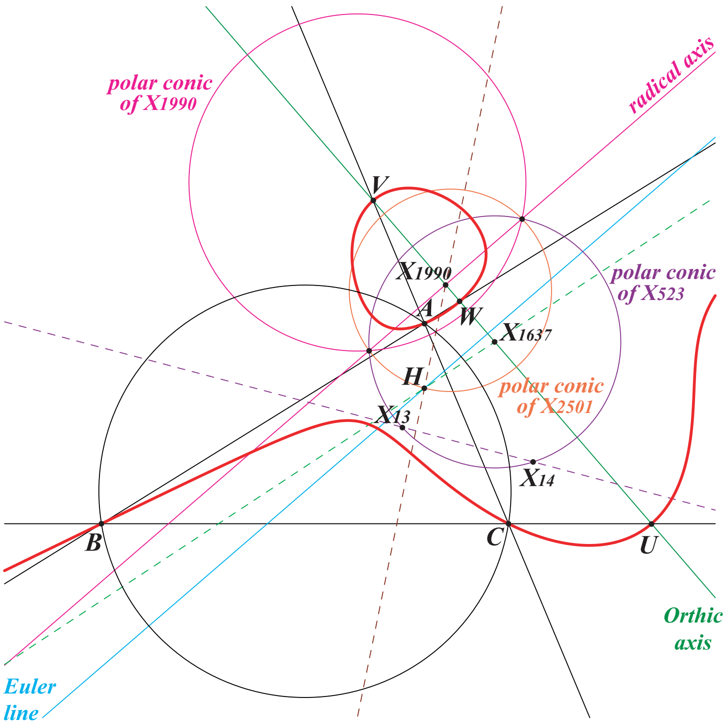

Figure 2.20: $n\mathcal{K}_0(X_{1990}, X_4)$, a non-pivotal cubic having a pencil of circular polar conics

2.5.2 Prehessians

When we apply the same technique we used for pivotal cubics, we obtain again a third degree equation in t which always factorizes into one first degree factor and one second degree factor. This shows that a $n\mathcal{K}_0$ has always one real prehessian and two other that can be real or not, distinct or not.

This depends of the sign of the discriminant

$$\Delta = (q^2r^2u^4 + r^2p^2v^4 + p^2q^2w^4) - 2pqr\left(pv^2w^2 + qw^2u^2 + ru^2v^2\right)$$

which rewrites under the form

$$\Delta = -p^2q^2r^2\left(\frac{u}{\sqrt{p}} + \frac{v}{\sqrt{q}} + \frac{w}{\sqrt{r}}\right)\left(-\frac{u}{\sqrt{p}} + \frac{v}{\sqrt{q}} + \frac{w}{\sqrt{r}}\right)$$
$$\left(\frac{u}{\sqrt{p}} - \frac{v}{\sqrt{q}} + \frac{w}{\sqrt{r}}\right)\left(\frac{u}{\sqrt{p}} + \frac{v}{\sqrt{q}} - \frac{w}{\sqrt{r}}\right)$$

showing that the sign of Δ depends on the position of the root P with respect to the different regions of the plane delimited by the trilinear polars of the square roots of the pole Ω. Recall that these square roots are real if and only if Ω lies inside ABC.

When Ω is the Lemoine point, the isoconjugation is isogonality and the square roots of K are the in/excenters of ABC. Δ is positive in the regions containing the vertices of ABC and negative in the regions containing the in/excenters of ABC. See figure 2.21.

We illustrate this with three isogonal focal $n\mathcal{K}_0$ with root on the orthic axis of ABC namely :

- the second Brocard cubic **K018** $= n\mathcal{K}_0(X_6, X_{523})$ which has always three distinct prehessians but one only is real ($\Delta < 0$). See figure 2.22.

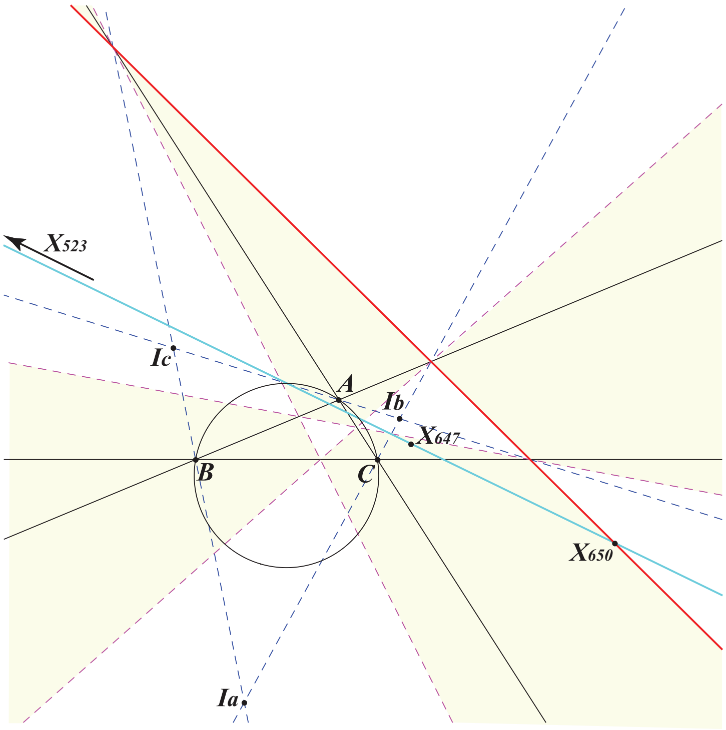

Figure 2.21: The sign of the discriminant Δ

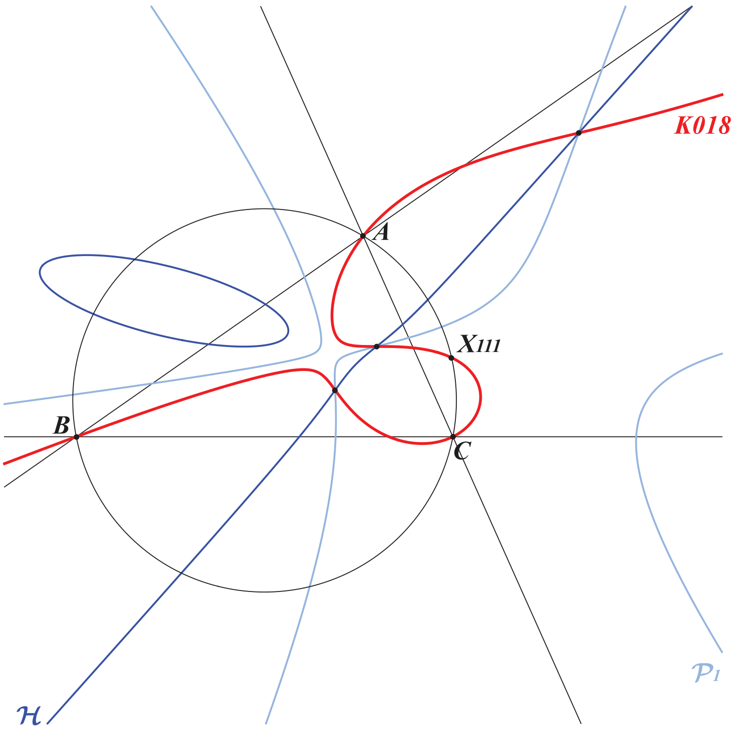

Figure 2.22: **K018** $= n\mathcal{K}_0(X_6, X_{523})$

- the third Brocard cubic **K019** $= n\mathcal{K}_0(X_6, X_{647})$ which has always three distinct real prehessians $(\Delta > 0)$. See figure 2.23.

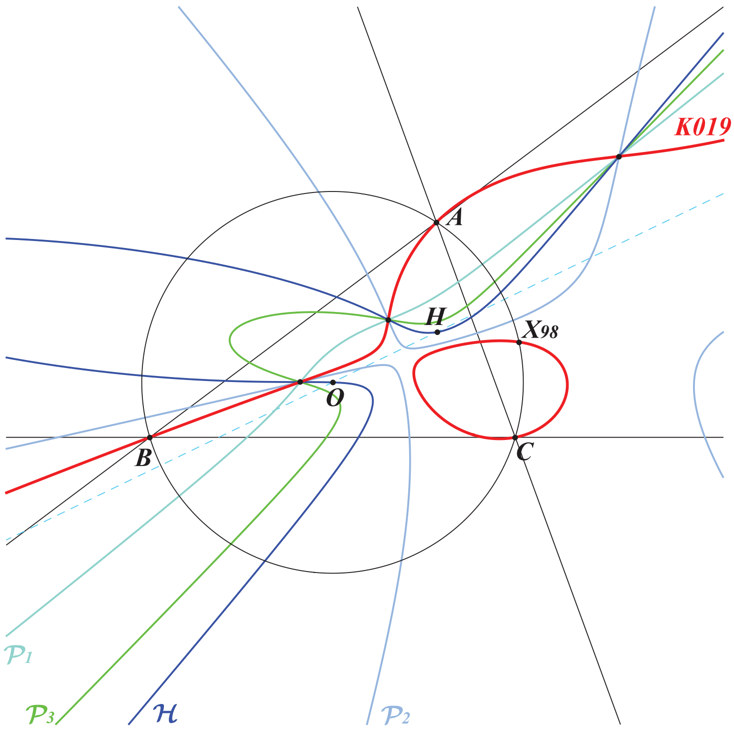

Figure 2.23: **K019** $= n\mathcal{K}_0(X_6, X_{647})$

- the Pelletier strophoid **K040** $= n\mathcal{K}_0(X_6, X_{650})$ which has always three real prehessians but two of them coincide $(\Delta = 0)$. See figure 2.24.

2.6 Polars and tripolars

In this section, we consider that $\mathcal{K}$ is the degenerate cubic which is the union of the sidelines of triangle ABC. All the definitions and results met in the previous sections are easily adapted and this gives a connection with all sorts of notions which are deeply related to the (projective) geometry of ABC.

2.6.1 With one extra point or line in the triangle plane

Let P be a point not lying on one sideline of ABC. With §2.1, we find the following results :

- The polar line of P in $\mathcal{K}$ is the trilinear polar of P with respect to ABC.
- The polar conic of P in $\mathcal{K}$ is the circum-conic with perspector P and center G/P.
- Thus, the trilinear polar of P is the polar line of P in the circum-conic with perspector P.
- Q lies on the circum-conic with perspector P if and only if P lies on the trilinear polar of Q.

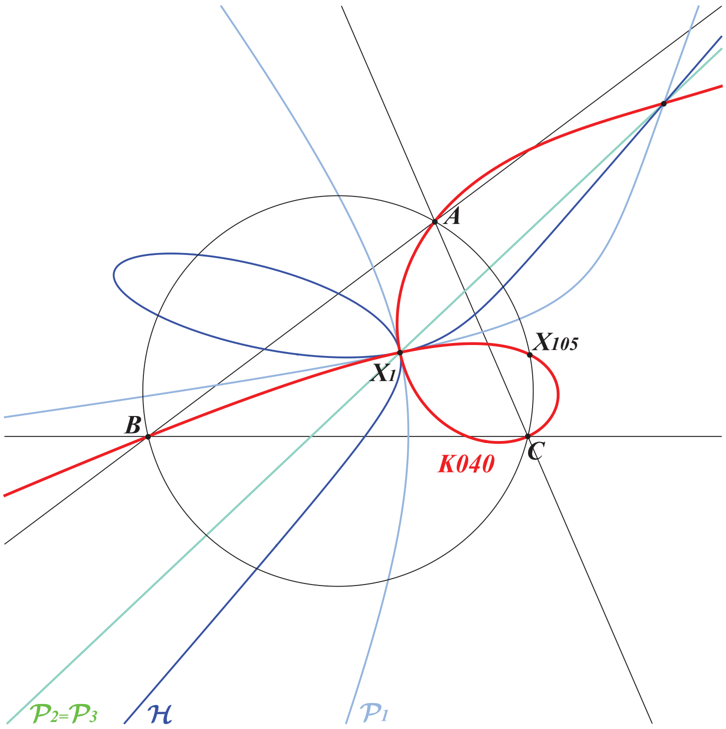

Figure 2.24: **K040** $= n\mathcal{K}_0(X_6, X_{650})$

Let $\mathcal{L}$ be a line which is not one sideline of ABC. From §2.3.2, we obtain :

- The poloconic of $\mathcal{L}$ in $\mathcal{K}$ is the inscribed conic with perspector P, the tripole of $\mathcal{L}$. Its center is **ct**P.
- The polar line in $\mathcal{K}$ of any point on $\mathcal{L}$ is therefore tangent to this inscribed conic.
- The polar conic of any point on the inscribed conic with perspector P is tangent to the trilinear polar of P.

2.6.2 With two extra points or lines in the triangle plane

Let P_1 and P_2 be two points not lying on one sideline of ABC.

Denote by $\mathcal{L}_1$, $\mathcal{L}_2$ their polar lines in $\mathcal{K}$ and by Γ_1, Γ_2 their polar conics in $\mathcal{K}$. $\mathcal{L}$ is the line P_1P_2 and Γ the circum-conic through P_1, P_2.

Recall that Γ_1, Γ_2 have in common A, B, C and a fourth point which is the trilinear pole of the line $\mathcal{L}$.

From §2.2.5, we see that the polar lines of P_2 in Γ_1 and P_1 in Γ_2 coincide. This line is called the **mixed trilinear polar** of P_1 and P_2. It is the line passing through the two poles of $\mathcal{L}$ in the conics Γ_1, Γ_2. The trilinear pole of this line is called cevian product or Ceva-point of P_1 and P_2.

In particular, if the cubic is not a stelloid, the mixed trilinear polar of the circular points at infinity is the orthic line of the cubic. This is the locus of points whose polar conics are rectangular hyperbolas. See chapter 5 for further informations. In our case, this orthic line is the orthic axis of ABC i.e. the locus of the perspectors of the rectangular circum-hyperbolas.

The mixed poloconic (see §2.3.5) of the lines $\mathcal{L}_1$, $\mathcal{L}_2$ is the conic which passes through the six feet of the cevian lines of P_1 and P_2. This we call the bicevian conic of P_1 and P_2 denoted by $\mathcal{C}(P_1, P_2)$.

Obviously, when $\mathcal{L}_1 = \mathcal{L}_2$, this conic is inscribed in ABC. In other words, any inscribed conic can be seen as the poloconic of the trilinear polar of its perspector with respect to $\mathcal{K}$ and any conic meeting the sidelines of ABC at six points forming two cevian triangles can be seen as the mixed poloconic of the trilinear polars of the corresponding perspectors.

For example, the Steiner in-ellipse is the poloconic of the line at infinity and the nine-point circle is the mixed poloconic of the line at infinity and the orthic axis.

Furthermore, for a given point P, there is one and only one point Q such that the mixed poloconic of the trilinear polars of P and Q is a circle : Q must be the cyclocevian conjugate of P. The locus of Q such that this mixed poloconic is a rectangular hyperbola is a circum-conic whose perspector is the point $S_B\, q(p + r) + S_C\, r(p + q)$ :: when $P = p : q : r$. For example, with $P = G$, we have $Q = K$ (Lemoine point), hence any rectangular hyperbola passing through the midpoints of ABC passes through the feet of the cevian lines of a point on the circumcircle (and through the circumcenter O).

Chapter 3

Central, oblique, axial isocubics Harmonic homologies

This chapter is devoted to isocubics invariant under symmetries and more generally under harmonic homologies.

3.1 Central $p\mathcal{K}$ isocubics

An isocubic $\mathcal{K}$ is said to be **central** if it is invariant under symmetry with respect to a point N called its **center**. Such a center is necessarily an inflexion point on $\mathcal{K}$. [1]

3.1.1 An involutive transformation

Let $\Lambda : M(p : q : r) \mapsto N(p(q + r - p) : q(r + p - q) : r(p + q - r))$ be the mapping which associates to each point M the center N of the circum-conic with perspector M. We have $N = G/M$ and we see that Λ is involutive. N is called the G-Ceva conjugate of M.

3.1.2 Theorem

There is a unique non-degenerate central $p\mathcal{K}$ invariant under a given Ω-isoconjugation which is not isotomic conjugation.
Moreover :

1. Its center is $N = \Lambda(\Omega)$ and its pivot is $P = h_{G,4}(N)$.
2. The three asymptotes are the lines through N and the midpoints of the sides of ABC.
3. The inflexional tangent at N is the line $N\Omega$.

Remarks :

1. $h_{G,4}(N)$ is the reflection of N^* in N. Equivalently, it is $\mathbf{a}N^*$.
2. Λ being involutive, there exists one and only one non-degenerate central $p\mathcal{K}$ with given center $N \neq G$ and not at infinity.

[1]Central cubics are called "cubiques de Chasles" by F. Gomes Teixeira.

3. N lies on $\mathcal{L}^\infty$ if and only if Ω lies on the inscribed Steiner ellipse. In this case, $P = \Omega$ and the cubic degenerates into three parallels through the vertices of triangle ABC.

3.1.3 Cubic curves from P-perspectivity

Paul Yiu [61] has studied a family of cubics he obtained using Floor van Lamoen's notion of P-perspectivity [60]. We quote his essential ideas, replacing P by Q in order to keep our own notations coherent.
Let Q be a point with homogeneous barycentric coordinates $(u : v : w)$. For each point X, the Q-traces of X are the intersections of the side lines of the reference triangle ABC with the lines through X parallel to AQ, BQ, CQ respectively. If X has barycentric coordinates $(x : y : z)$, the Q-trace triangle is perspective with the reference triangle if and only if

$$\sum_{\text{cyclic}} (v + w - u)\, x \left[w(u + v)y^2 - v(w + u)z^2\right] = 0.$$

This defines a central $p\mathcal{K}$ denoted by $p\mathcal{K}_c(Q)$ with :

- pole $\Omega(u(v+w) : v(w+u) : w(u+v)) = \mathbf{ct}Q$, the center of the iconic with prospector Q,
- pivot $P(v + w - u : w + u - v : u + v - w) = \mathbf{a}Q$,
- center $N(v + w : w + u : u + v) = \mathbf{c}Q$.

Its asymptotes are the parallels to the cevian lines of Q through N and they pass through the midpoints of the sides of ABC.

Proposition

Every non-degenerate central $p\mathcal{K}$ is a $p\mathcal{K}_c(Q)$ for some Q.
Proof : $Q = \mathbf{ta}\Omega$.

Remark :

The locus of the perspector of Q-trace and reference triangles is the isotomic $p\mathcal{K}$ with pivot $\mathbf{t}Q$.
Its equation is :

$$\sum_{\text{cyclic}} vwx(y^2 - z^2) = 0$$

For example, when $Q = H$, the central $p\mathcal{K}$ is the Darboux cubic **K004** and the isotomic $p\mathcal{K}$ above is the Lucas cubic **K007**.

Proposition

Conversely, it is easy to obtain a central $p\mathcal{K}$ from an isotomic $p\mathcal{K}$.
Let M be a point on the isotomic $p\mathcal{K}$ with pivot $\mathbf{t}Q$, with traces A_M, B_M and C_M. The lines through A_M, B_M, C_M parallel to AQ, BQ, CQ respectively concur at a point Z. As M traverses the isotomic $p\mathcal{K}$, Z traverses a central $p\mathcal{K}$.
The inflexional tangent at its center is parallel to $Q\mathbf{t}Q$. We can observe that two isotomic conjugates M and $\mathbf{t}M$ on the isotomic $p\mathcal{K}$ lead to two points Z and Z' symmetric about the center on the central $p\mathcal{K}$.

3.2 Some examples of central $p\mathcal{K}$ isocubics

3.2.1 The Darboux cubic K004 : the only isogonal central $p\mathcal{K}$

When $\Omega = K$, we find the well-known Darboux cubic with center O and pivot L. Its asymptotes are the perpendicular bisectors of ABC. See [9] for details.

3.2.2 The isotomic central $p\mathcal{K}$

When $\Omega = G$, we find a degenerate cubic into the three medians of ABC.

3.2.3 The Fermat cubics K046a, K046b : the two central $p\mathcal{K}_{60}^{++}$

There are only two central $p\mathcal{K}$ which are $p\mathcal{K}_{60}^{++}$. We call them the Fermat cubics and they will be seen in §6.6.4 below.

3.2.4 Other remarkable central $p\mathcal{K}$

The table below shows a small selection of central $p\mathcal{K}$ with pole Ω, pivot P and center N are the pole of isoconjugation, the pivot, the center of symmetry respectively.

Notations :

- $N = X_5$ = nine point center
- $Ge = X_7$ = Gergonne point
- $Na = X_8$ = Nagel point
- $Mi = X_9$ = Mittenpunkt
- $Sp = X_{10}$ = Spieker center

Q	Ω	P	N	P^*	other centers	cubic
I	X_{37}	Na	Sp	X_{65}	$X_{4,40,72}$, see note 2	**K033**
O	X_{216}	H	N	X_d	$X_{52,68,155}$, see note 1	**K044**
H	K	L	O	X_{64}	$X_{1,40,84}$	**K004**
K	X_{39}	$\mathbf{t}H$	X_{141}	Xb	$X_{66,159}, X_c$	**K140**
Ge	I	X_{144}	Mi	X_a	X_{366}	**K202**
Na	Mi	X_{145}	I	X_e	X_{188}	**K201**
Sp	X_{1213}	I	X_{1125}	X_i	X_{596}	
X_{13}	X_{396}	X_{616}	X_{618}			**K046a**
X_{14}	X_{395}	X_{617}	X_{619}			**K046b**
X_{66}	X_{32}	$\mathbf{a}X_{66}$	X_{206}	X_f	K, X_{159}	**K161**
X_{67}	X_{187}	$\mathbf{a}X_{67}$	X_h		see note 3	**K042**
X_{69}	O	X_{193}	K	X_g		
X_{74}	X_{3003}	X_{146}	X_{113}		$X_{265,399,1986,2935}$	**K255**

Unlisted centers in [38, 39] :

- $X_a = \left[a/[(b-c)^2 + a(2b+2c-3a)]\right] = E_{2326}$
- $X_b = \left[a^2(b^2+c^2)/S_A\right] = E_{2334}$
- $X_c = \left[a^2(b^2+c^2)(b^4+c^4-a^4)\right]$ on the line OK
- $X_d = \left[a^2 S_A^2[(b^2-c^2)^2 - a^2(b^2+c^2)]\right] = E_{593}$
- $X_e = [a(b+c-a)/(b+c-3a)] = E_{2327}$
- $X_f = \left[a^4/[(b^4-c^4)^2 + a^4(2b^4+2c^4-3a^4)]\right]$
- $X_g = \left[a^2 S_A/(b^2+c^2-3a^2)\right] = E_{2329}$
- $X_h = \left[a^2(b^2+c^2-2a^2)(b^4+c^4-a^4-b^2c^2)\right] = E_{406}$ is the midpoint of X_6, X_{110} on the Jerabek hyperbola
- $X_i = (b+c)(b+c+2a)/a$

Notes :

1. **K044** is called the Euler central cubic. See figure 3.1. It contains 9 familiar points :
   - the feet of the altitudes since its pivot is H.
   - the centers of circles HBC, HCA, HAB which are the reflections about X_5 of the feet of the altitudes.
   - the reflections about X_5 of A, B, C, these points on the perpendicular bisectors.

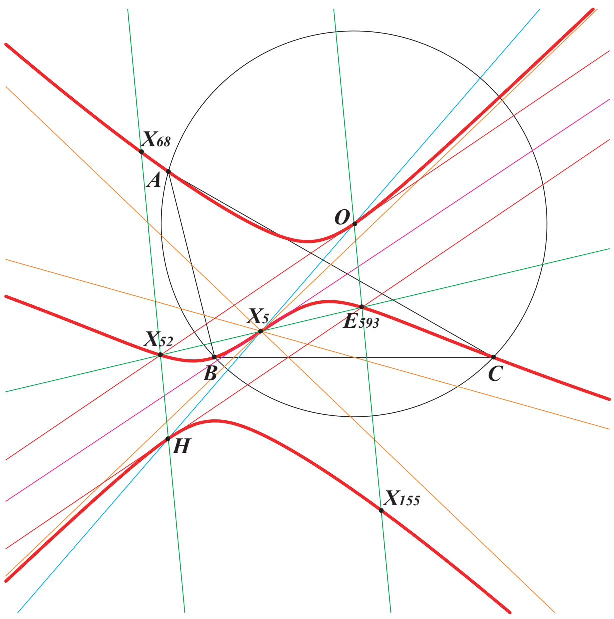

Figure 3.1: **K044** the Euler central cubic

2. This cubic **K033** is called the Spieker central cubic. It is obtained with $Q = I$ in §3.1.3 : for any point M on the curve, the parallels through M to AI, BI, CI meet ABC sidelines at three points forming a triangle perspective with ABC. Furthermore, the perspector lies on **K034** the isotomic $p\mathcal{K}$ with pivot $X_{75} = \mathbf{t}I$ called Spieker perspector cubic passing through $X_{1,2,7,8,63,75,92,280,347}, \ldots$

3. This cubic **K042** is called the Droussent central cubic. It is obtained with $Q = X_{67}$ in §3.1.3. This case is particularly interesting since the perspector as seen above lies on the Droussent cubic. See figure 3.2.

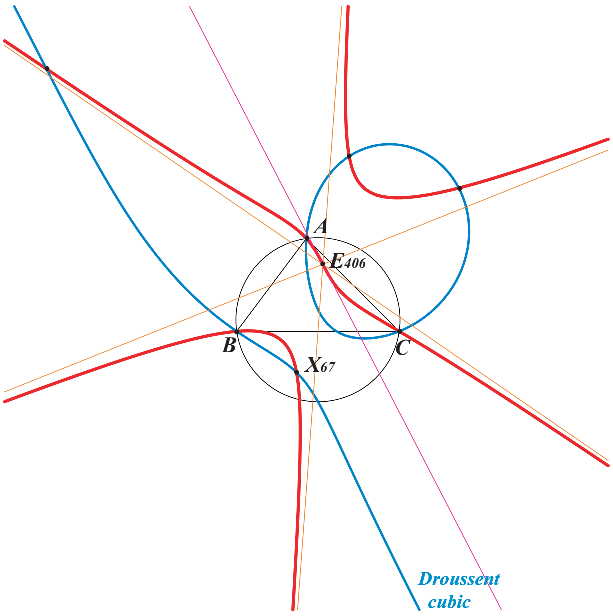

Figure 3.2: **K042** the Droussent central cubic

3.3 Central $n\mathcal{K}$ isocubics

We seek central $n\mathcal{K}$ isocubics knowing either the pole Ω of the isoconjugation, the center N or the root P of the cubic.
We denote by C_Ω the circum-conic with perspector Ω i.e. the isoconjugate of $\mathcal{L}^\infty$. Remember that C_Ω is a parabola, an ellipse, a hyperbola if and only if Ω lies on, inside, outside the inscribed Steiner ellipse respectively.

3.3.1 Theorem 1 : the pole Ω is given

For a given pole Ω,

- the locus of N is C_Ω.
- one asymptote (always real) of the cubic is the line NN^* ($N^* = \Omega$-isoconjugate of N) whose trilinear pole is denoted by Z_Ω.
- the remaining two asymptotic directions are those of C_Ω.
- the root P is $\mathbf{ct}Z_\Omega$.

Remarks :

1. The two other asymptotes of the cubic can be real or imaginary, distinct or not, according to the position of Ω with respect to the Steiner inscribed ellipse.

   More precisely, $n\mathcal{K}$ has :

   - one real asymptote if and only if Ω is inside the Steiner inscribed ellipse i.e. if and only if C_Ω is an ellipse. See §3.4.3 for example.
   - three real concurrent asymptotes if and only if Ω is outside the Steiner inscribed ellipse i.e. if and only if C_Ω is a hyperbola. See examples 1 and 2 in §3.4.1.
   - three asymptotes, two of them being parallel, if and only if Ω lies on the Steiner inscribed ellipse i.e. if and only if C_Ω is a parabola. In this case, the two parallel asymptotes are not always real and, therefore, we can find two different types of non-degenerate cubics with :
     - three real asymptotes, two of them being parallel, one real and two imaginary inflexion points, one node at infinity.
     - one real and two imaginary asymptotes, three real inflexion points, one node at infinity. See example 3 in §3.4.1.

2. $n\mathcal{K}$ can degenerate in several different ways : for instance the union of a hyperbola and one of its asymptotes, or the union of a parabola and the line which is its isoconjugate, or the union of three lines, two of them being parallel, etc.

3.3.2 Theorem 2 : the center N is given

For a given center N,

- the locus of Ω is $\mathbb{P}(N)$.
- the locus of the root P is $\mathbb{P}(\mathbf{ta}N)$.
- $\mathbb{P}(P)$ intersects ABC sidelines at U, V, W (which are on the cubic) such that the line UVW envelopes the inscribed conic centered at N. Moreover U, V, W are the symmetrics (with respect to N) of the Ω-isoconjugates A'', B'', C'' of the symmetrics A', B', C' of A, B, C.

3.3.3 Theorem 3 : the root P is given

- For a given root $P = (u : v : w)$, $\mathbb{P}(P)$ intersects the sidelines of triangle ABC at U, V, W.
  - the locus of the center N is $\delta_P = \mathbb{P}(\mathbf{ta}P)$.
  - the center N being chosen on δ_P, let A', B', C', U', V', W' be the reflections of A ,B, C, U, V, W about N. The pole Ω is the barycentric product of A' and U', B' and V', C' and W'. It lies on the trilinear polar of the point $[(v-w)/(v+w-u)]$ (isotomic conjugate of the point where $\mathbb{P}(\mathbf{c}P)$ and $\mathbb{P}(\mathbf{a}P)$ meet).
- It can be seen that all central $n\mathcal{K}s$ with a given root $P = (u : v : w)$ form a pencil of cubics passing through A, B, C, U, V, W, the point at infinity of δ_P, having a common real asymptote.

This pencil contains several particular cubics (any two of them generating the pencil) :

– if the center is $N = [u(v-w)(v+w-u)]$ [2] (intersection of δ_P and $\mathbb{P}(\mathbf{t}P)$), the cubic is the union of δ_P itself and the circum-conic which is its isoconjugate, this conic being centered at N and passing through A' ,B', C'. The pole is $\Omega = [u(v-w)^2(v+w-u)]$.
– if the center is at infinity, we obtain a cubic with a flex at infinity (Newton trident type) and $\Omega = [(v-w)^2]$ is a point on the inscribed Steiner ellipse.
– with $\Omega = [(v-w)(v+w-2u)/(v+w-u)]$ and $N = [(v+w-2u)/(v+w-u)]$, we find the only $n\mathcal{K}_0$ of the pencil.

For example, all the central $n\mathcal{K}s$ with root at X_{525} are centered on the Euler line and have the Euler line as common real asymptote. Their poles Ω lie on the line through X_{30}, X_{1990}, X_{3163}, X_{3284}. See Figure 3.3.

Those with root at X_{647} are centered on the Brocard line.

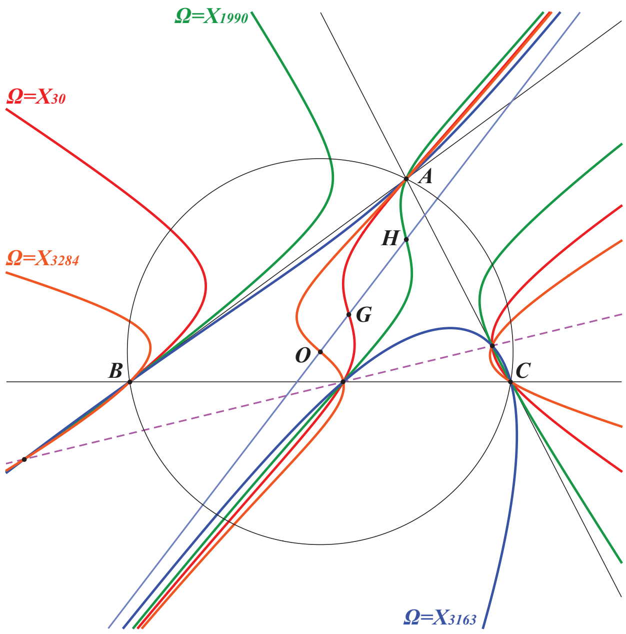

Figure 3.3: The Euler pencil of central $n\mathcal{K}s$

3.3.4 Proof of theorems 1 and 2 :

Starting from the equation of $n\mathcal{K}$ seen in §1.5 and from the equation $pyz+qzx+rxy=0$ of C_Ω, we express that the polar lines of the points at infinity of ABC sidelines and the cubic pass through the center $N(x_o : y_o : z_o)$.
We obtain the condition :

$$\mathcal{M}\begin{pmatrix} u \\ v \\ w \\ k \end{pmatrix} = 0$$

[2]This point is the perspector of the circum-conic which passes through P and $\mathbf{a}P$.

where

$$\mathcal{M} = \begin{pmatrix} (q+r)x_o & p(y_o-2z_o) & p(z_o-2y_o) & -x_o \\ q(x_o-2z_o) & (r+p)y_o & q(z_o-2x_o) & -y_o \\ r(x_o-2y_o) & r(y_o-2x_o) & (p+q)z_o & -z_o \\ x_o(ry_o^2+qz_o^2) & y_o(pz_o^2+rx_o^2) & z_o(qx_o^2+py_o^2) & x_oy_oz_o \end{pmatrix}$$

We find

$$\det\mathcal{M} = 2(x_o+y_o+z_o)(py_oz_o+qz_ox_o+rx_oy_o)\left(\sum_{\text{cyclic}} x_o^2(y_o+z_o-x_o)qr\right)$$

which leads to examine three cases :

(1) : $x_o+y_o+z_o=0$ which gives $n\mathcal{K} = d_\infty \cup C_\Omega$.

(2) : $x_o^2(y_o+z_o-x_o)qr+\cdots=0$ which is the condition for which the line Ω-isoconjugate of the circum-conic centered at N passes through N. In this situation, we get degenerate cubics into a line through N and its Ω-isoconjugate conic which is centered at N. See §3.3.3.

(3) : the non-degenerate cubics are therefore obtained when $N \in C_\Omega$.

Now, let us parametrize C_Ω by :

$$x_o = \frac{p}{\beta-\gamma}; \quad y_o = \frac{q}{\gamma-\alpha}; \quad z_o = \frac{r}{\alpha-\beta}$$

where $(\beta-\gamma : \gamma-\alpha : \alpha-\beta)$ is the point at infinity N^* i.e. the line NN^* is an asymptote of $n\mathcal{K}$.

From this, we get :

$$u = (\alpha-\beta))(\gamma-\alpha)(\beta+\gamma-2\alpha)p - (\beta-\gamma)^2[(\alpha-\beta)q-(\gamma-\alpha)r],$$

v and w similarly, and :

$$k = 2\sum_{\text{cyclic}} (\beta-\gamma)^2(\beta+\gamma-2\alpha)\, qr$$

This shows that the two remaining asymptotic directions are those of C_Ω.

Remark 1: this central cubic is a $n\mathcal{K}_0$ if and only if N^* is one of the three infinite points of the cubic $p\mathcal{K}(\Omega,\Omega)$. See §1.4.2.

Remark 2: if we denote by $u:v:w$ the coordinates of N^* (assuming $u+v+w=0$) the equation of the central $n\mathcal{K}$ rewrites under the form :

$$\sum_{\text{cyclic}} \left[vw(v-w)p - u^2(wq-vr)\right] x(ry^2+qz^2) + 2\left[\sum_{\text{cyclic}} u^2(v-w)qr\right] xyz = 0.$$

3.3.5 Construction of a central $n\mathcal{K}$

First, choose Ω and N according to one of the three theorems above, then

1. draw the symmetrics A', B', C' of A, B, C with respect to N.
2. draw the Ω-isoconjugates A'', B'', C'' of A', B', C'.

3. draw the symmetrics U, V, W of A'', B'', C'' with respect to N (they are collinear and on ABC sidelines). We can notice that the lines AA'', BB'', CC'', UA', VB', CW', NN^* are all parallel which yields the construction of the polar conic of N^* centered at N passing through the midpoints of AA'', BB'', CC'', UA', VB', CW'.

4. draw the trilinear pole P of UVW (it is the root of the cubic).

5. intersect the lines NA and UVW at N_a then define N_b, N_c similarly.

6. Now, let $\mathcal{J}$ be the involution on UVW which swaps U and N_a, V and N_b, W and N_c.

   • for any point m on UVW, construct its image m' under $\mathcal{J}$ and draw the lines $\ell = Nm$ and $\ell' = Nm'$ (this defines an involution on the pencil of lines through N).

   • let γ and γ' be the conics which are the Ω-isoconjugates of ℓ and ℓ' resp.

   • intersect ℓ and γ' at X and Y, ℓ' and γ at X^* and Y^*. Those four points are on the cubic, two by two symmetric about N, two by two Ω-isoconjugates.

7. At last, draw the line NN^* which is one real asymptote of the cubic and, the parallels at N to the asymptotes of C_Ω when it is a hyperbola (this cannot be done when C_Ω is an ellipse) which are the remaining two asymptotes. Let us remark that C_Ω passes through N, its center being G/Ω.

   When C_Ω is a parabola, the two parallel asymptotes (whose union is the polar conic of the point at infinity of the parabola axis) are less easy to draw, and remember that they are not always real : one solution is to draw the tangents to the hyperbola defined tangentialy with the six lines AU, BV, CW, $A'A''$, $B'B''$, $C'C''$ which are parallel to the axis of the parabola.

8. The inflexional tangent at N is easy to draw : it is the homologue of the line NN^* under the involution on the pencil of lines through N as defined above. In a simpler way, it is also the tangent at N to the circum-conic through N and N^*.

9. The two lines (real or not) through N and the fixed points of the involution $\mathcal{J}$ meet the polar conic of N^* at four points (real or not) where the tangents are parallel to the real asymptote. The quadrilateral formed with those four points is a parallelogram centered at N whose sidelines are two by two parallel to the asymptotes other than the real one NN^*.

   When the cubic is isogonal (see below), those four points are the centers of inversion of the cubic and the two lines through N are perpendicular. The inversive images of the inflexional tangent are the oscultator circles at the four points.

3.4 Some examples of central $n\mathcal{K}$ isocubics

3.4.1 Selected examples

Example 1 : K068

When $N = G$ and $\Omega = X_{523}$ (point at infinity of orthic axis), we find a $n\mathcal{K}_0$ with equation :

$$\sum_{\text{cyclic}} (b^2 + c^2 - 2a^2)\, x\, [(a^2 - b^2)y^2 + (c^2 - a^2)z^2] = 0$$

Its root is $P = X_{524}$ and C_Ω is the Kiepert hyperbola. One of its asymptotes is the line GK and the others are the parallels at G to those of the Kiepert hyperbola. The inflexional tangent at G passes through X_{99}. See figure 3.4.

More generally, each central non-degenerate $n\mathcal{K}$ centered at G is a $n\mathcal{K}_0$ with pole and root at infinity. It always has three real asymptotes.

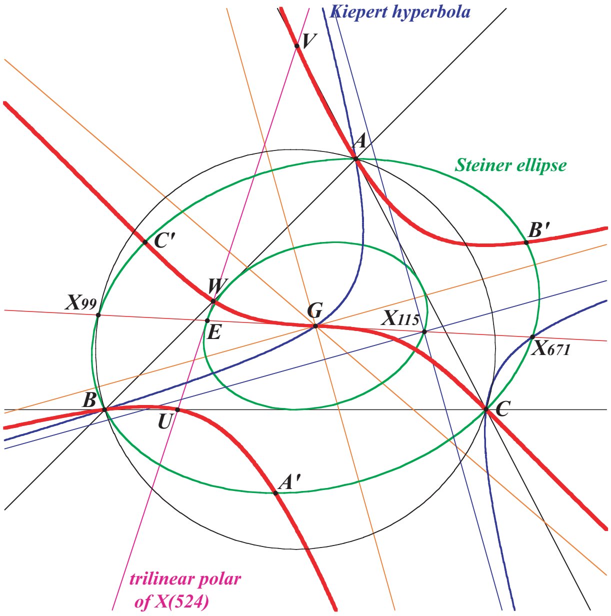

Figure 3.4: **K068** a central $n\mathcal{K}$ with center G and pole X_{523}

Example 2 : K069

When $N = O$ and $\Omega = X_{647} = [a^2(b^2 - c^2)S_A]$ (isogonal conjugate of trilinear pole of Euler line, a point on orthic axis), we find a central $n\mathcal{K}$ with root :

$$P = \left[\left((b^2 - c^2)^2 - a^2(b^2 + c^2 - 2a^2)\right) S_A\right]$$

and C_Ω is the Jerabek hyperbola. One of its asymptotes is the perpendicular at O to the Euler line and the others are the parallels at O to those of the Jerabek hyperbola. See figure 3.5.

Example 3 : K036 Tixier central cubic

When $N = X_{476}$ (Tixier point) and $\Omega = X_{115}$ (center of Kiepert hyperbola, lying on the inscribed Steiner ellipse), we find a central $n\mathcal{K}$ with two asymptotes parallel to the axis of the parabola C_Ω with equation :

$$\sum_{\text{cyclic}} (b^2 - c^2)^2 \, y \, z = 0$$

i.e. two asymptotes (not necessarily real) perpendicular to the Euler line. See figure 3.6.

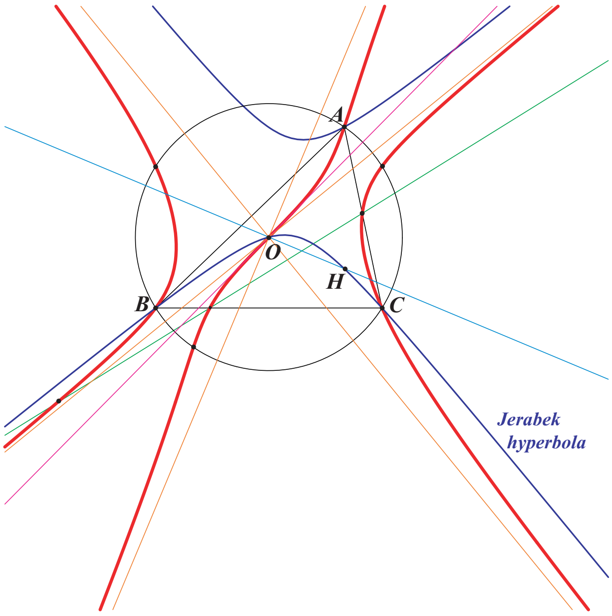

Figure 3.5: **K069** a central $n\mathcal{K}$ with center O and pole X_{647}

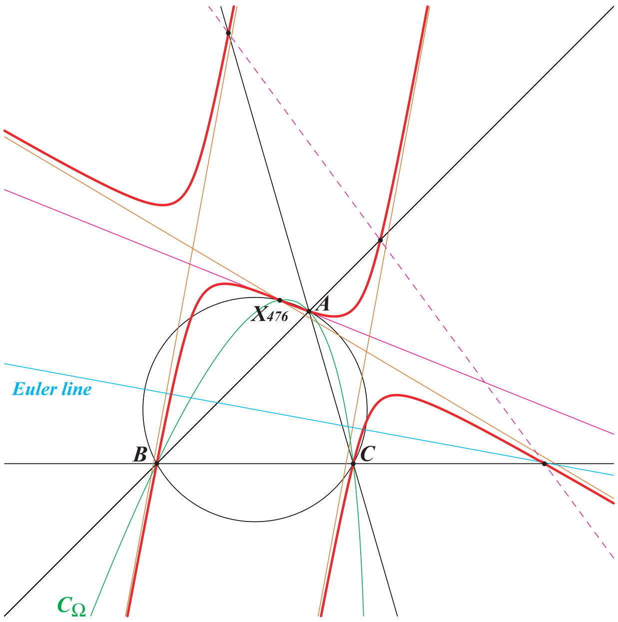

Figure 3.6: **K036** Tixier central $n\mathcal{K}$ with center X_{476} and pole X_{115}

3.4.2 Isogonal central $n\mathcal{K}$ cubics. K084

Theorem 1 above with $\Omega = K$ shows that all isogonal central $n\mathcal{K}$ are circular [3] with a center N on the circumcircle [4], with a real asymptote perpendicular at N to the Simson line of N. In this case, the singular focus is obviously the center.

[3]The asymptotes of the circumcircle are the isotrope lines of O which shows that the cubic must pass through the circular points at infinity.

[4]The circum-conic with perspector K is the circumcircle.

Remark :

The cubic can degenerate into a circle and a line through the center of the circle. This happens when the center of the cubic is the "second" intersection of a bisector of ABC with the circumcircle.

The figure 3.7 shows the Steiner central isogonal cubic with center X_{99}, the Steiner point.

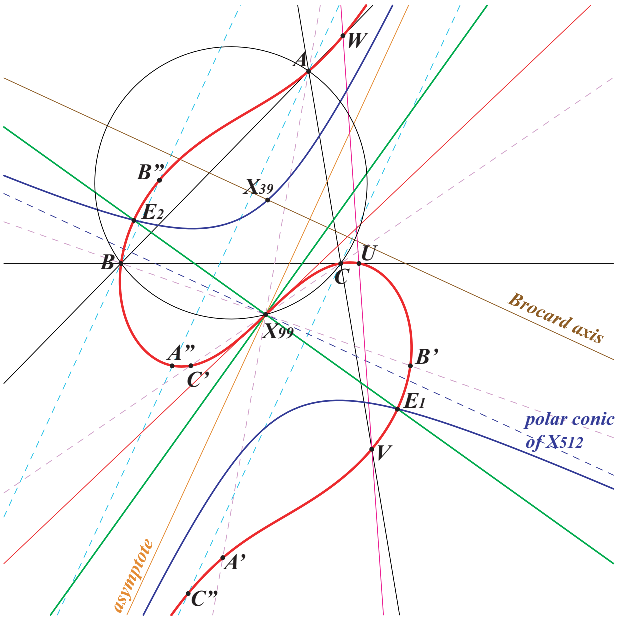

Figure 3.7: **K084** Steiner central isogonal cubic

3.4.3 Isotomic central $n\mathcal{K}$ cubics. K087

Now with $\Omega = G$, we find that all isotomic central $n\mathcal{K}$ are centered on the Steiner circum-ellipse and they always have only one real asymptote.

The most remarkable is obtained with a center $N = X_{99}$ (Steiner point), the real asymptote being the perpendicular at this point to the Euler line. Its root is :

$$R = (R_a : R_b : R_c) = \left((b^2 + c^2 - 2a^2)[(b^2 - c^2)^2 - b^2c^2 + 2a^2 S_A] : \cdots : \ldots\right)$$

and its equation :

$$\sum_{\text{cyclic}} R_a\, x\,(y^2 + z^2) - 2 \prod_{\text{cyclic}} (b^2 + c^2 - 2a^2)\, x = 0.$$

In other words, there is only one point N such as there are a central isogonal $n\mathcal{K}$ and a central isotomic $n\mathcal{K}$ with the same center N : N must be the Steiner point.

The figure 3.8 shows the Steiner central isotomic cubic with center the Steiner point.

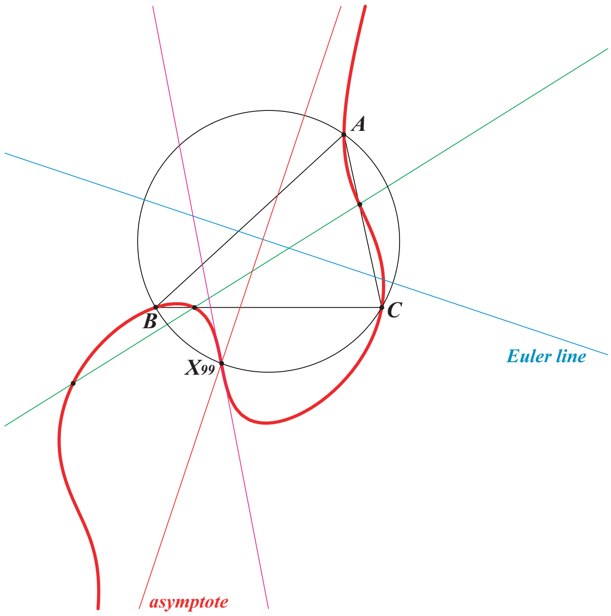

Figure 3.8: **K087** Steiner central isotomic cubic

3.5 Oblique and axial $p\mathcal{K}$ isocubics

3.5.1 Oblique and axial symmetries

Let L be a (finite) line and F a point at infinity which is not that of L. The oblique symmetry $S_{L,F}$ with axis L and center F maps a point M in the plane to the point M' such that F, M, M' are collinear and L bisects MM'. When F is the infinite point of the direction perpendicular to L, $S_{L,F}$ is said to be an axial (or orthogonal) symmetry.

We study the isocubics with pole $\Omega = p : q : r$ which are invariant under such oblique or axial symmetry, in which case the cubic will be called an **oblique cubic** or an **axial cubic**.

Lemma 1 *If an isocubic meeting $\mathcal{L}^\infty$ at the (real) point F is oblique then F must be a flex on the cubic and the center of the symmetry.*

Lemma 2 *In such case, the polar conic of F in the cubic must degenerate into the real asymptote (i.e. the tangent at F to the cubic) and the harmonic polar of F which must be the axis of the symmetry.*

3.5.2 Oblique $p\mathcal{K}$

Theorem 1 *Given the pole Ω and one real infinite point F on the pivotal isocubic $p\mathcal{K}$, there are at most three (one always real) pivots P_i, $i \in \{1, 2, 3\}$, such that $p\mathcal{K}(\Omega, P)$ is an oblique cubic.*

Obviously these pivots must lie on the line FF^*. Note that all the $p\mathcal{K}(\Omega, P)$ with P on FF^* form a pencil of cubics.

Let us denote by $\mathcal{A}_i$ and $\mathcal{L}_i$ the asymptote and the axis of symmetry of $p\mathcal{K}(\Omega, P_i)$. Since the polar conics of F in the $p\mathcal{K}$ with pivot on FF^* form a pencil of conics, they must have four common points, one of them being F, the other F_1, F_2, F_3. This leads to the following theorem.

Theorem 2 *The asymptote $\mathcal{A}_i$ of $p\mathcal{K}(\Omega, P_i)$ contains the intersection F_i of the axes $\mathcal{L}_j$ and $\mathcal{L}_k$ of $p\mathcal{K}(\Omega, P_j)$ and $p\mathcal{K}(\Omega, P_k)$.*

The locus of the centers of the polar conics of F is a conic which must pass through the midpoints of the four points above but, since F lies at infinity, this conic is a parabola.

3.5.3 Axial $p\mathcal{K}$

Generally, for a given pole Ω, the pencil of polar conics of F contains one and only one rectangular hyperbola which is not degenerate hence we cannot find an axial $p\mathcal{K}$. But, for a given infinite point F, there are two poles Ω_1, Ω_2 such that these polar conics are rectangular hyperbolas for any pivot on the line FF^*. Recall that F^* is the $\Omega-$isoconjugate of F.

Ω_1 is the barycentric square F^2 of F (F^2 lies on the Seiner in-ellipse) which leads to a degenerate $p\mathcal{K}$ into the cevians of F. These three parallels give a trivial axial $p\mathcal{K}$.

With $F = u : v : w$, the other pole is the point $\Omega_2 = f(F)$ where

$$f(F) = \frac{S_B\, v - S_C\, w}{c^2v^2 - b^2w^2} : : \quad \sim \quad (S_B\, v - S_C\, w)(a^2w^2 - c^2u^2)(b^2u^2 - a^2v^2) : : .$$

This point $f(F)$ lies on the line KF^2 and can be constructed as the barycentric product of the infinite point of the direction perpendicular to F and the trilinear pole of the line KF^2 (this latter point on the circumcircle). This point $f(F)$ is also the barycentric product of the orthocenter H and the pole of the line HF in the diagonal rectangular hyperbola that passes through F and the in/excenters of ABC.

Hence the following theorem :

Theorem 3 *For a given infinite point F, the cubic $p\mathcal{K}(f(F), P)$ is such that the polar conic of F is a rectangular hyperbola for any pivot P on the line FF^*.*

These polar conics belong to a same pencil of rectangular hyperbolas having parallel asymptotes and passing through two other (real or not) points. We already know that the locus of their centers is a parabola which must degenerate into $\mathcal{L}^\infty$ and another line. To draw this line, it is enough to remark that the pencil contains two particular hyperbolas $\mathcal{H}_c$ and $\mathcal{H}_d$ with centers O_c and O_d.

– $\mathcal{H}_c$ is the rectangular circum-hyperbola passing through F, F^*,

– $\mathcal{H}_d$ is the diagonal rectangular hyperbola passing through F, the vertices of the anticevian triangle of F and its orthocenter[5]. Its center O_d is the trilinear pole of the line $Ff(F)$. It lies on the circumcircle of ABC, on the circum-conic with perspector $f(F)$ and is the antipode of F^* on this conic.

Actually, F^* and O_d are two vertices of the circum-conic with perspector $f(F)$. Note that $\mathcal{H}_d$ also contains the (real or not) fixed points of the isoconjugation with pole $f(F)$. See figure 3.9.

From this we deduce that all the rectangular hyperbolas of the pencil are centered on the line O_dO_c. Among them there are three degenerate hyperbolas :

– one into $\mathcal{L}^\infty$ and the line through the two finite (real or not) common points of the pencil which leads to an oblique cubic,

[5]Anticevian triangle of an infinite point F : the line AF meets BC at a point whose harmonic conjugate in B, C is A'. The line AA' is a sideline of the sought triangle. Similarly, draw the lines BB' and CC'.

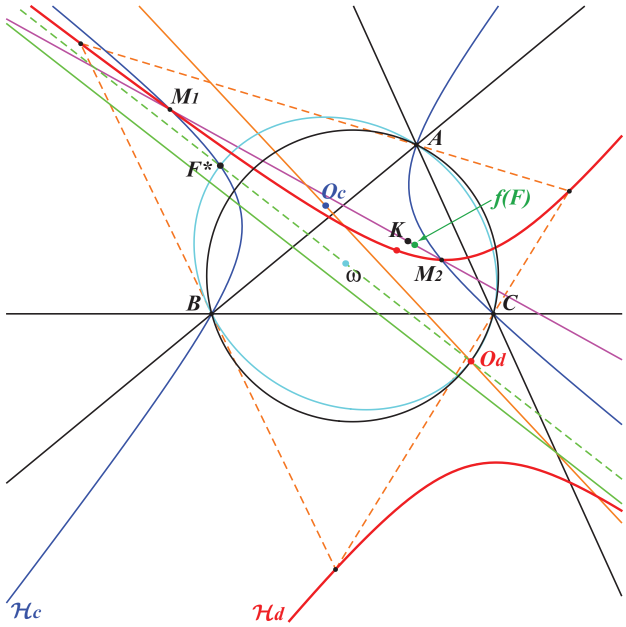

Figure 3.9: The two hyperbolas $\mathcal{H}_c$, $\mathcal{H}_d$

– two pairs of perpendicular lines, each line passing through one finite point and one infinite point of the four basis points of the pencil of polar conics.

This shows that there are two corresponding possible pivots on FF^* although they might not be always real. This will be the case when $\mathcal{H}_c$ and $\mathcal{H}_d$ do intersect. Hence we obtain :

Theorem 4 *For a given infinite point F, there are two (not necessarily real) $p\mathcal{K}$ which are invariant under the axial symmetry with axis a direction perpendicular to F.*

Let now suppose that $\mathcal{H}_c$ and $\mathcal{H}_d$ have two real (finite) common points M_1 and M_2. These points lie on the line $Kf(F)$ – which is the polar of G in $\mathcal{H}_d$ – and on the Thomson cubic **K002** hence M_1 and M_2 are two $G-$Ceva conjugate points since K is the isopivot of **K002**.

The perpendiculars $\mathcal{A}_1$, $\mathcal{A}_2$ at M_1, M_2 to FF^* meet the line O_cO_d at O_1, O_2 which are the centers of the degenerate polar conics. This gives the axes $\mathcal{A}_1$, $\mathcal{A}_2$ and the real asymptotes (the perpendiculars at O_1, O_2 to these axes) of the sought cubics. The reflections of F^* in these axes are the sought pivots P_1, P_2 of the cubics.

Clearly, these two pivots are symmetric with respect to the center ω of the circum-conic with perspector $f(F)$. See figure 3.10.

3.5.4 Isotomic and isogonal axial $p\mathcal{K}$

The transformation f above is not defined for A, B, C, H and the in/excenters. It maps any infinite point F to a point which lies on the quintic $\mathcal{Q}$ = **Q053** which has equation

$$\sum_{\text{cyclic}} a^4[a^2(y-z) - 3(b^2-c^2)x]y^2z^2 = 0.$$

$\mathcal{Q}$ contains the following points :

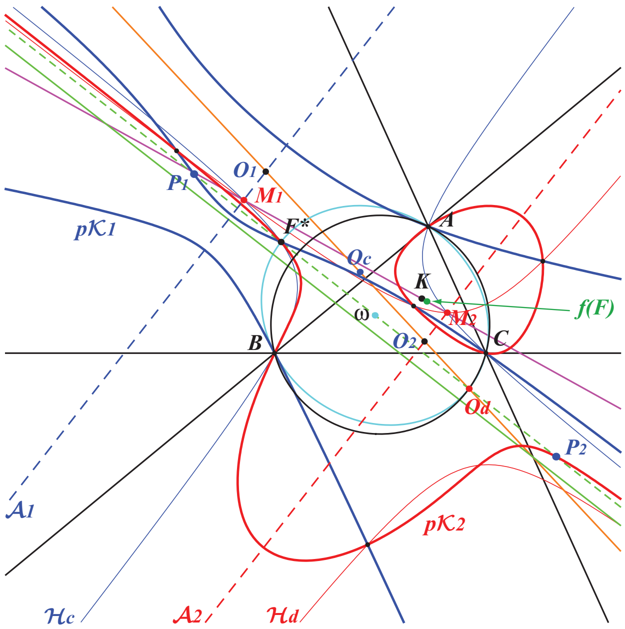

Figure 3.10: Construction of the pivots of the axial $p\mathcal{K}$

- A, B, C (which are double),
- the midpoints of ABC (images of the infinite points of the altitudes),
- K (which is triple),
- X_{1989}, the image of X_{30}, the infinite point of the Euler line,

Figure 3.11 shows the two axial $p\mathcal{K}$ obtained with $F = X_{30}$, one of them being **K528**. Since $\mathcal{Q}$ does not contain the centroid G, we see that there is no isotomic axial $p\mathcal{K}$.

f maps any point on the McCay cubic to the Lemoine point K and, in particular, its infinite points which are also the infinite points of the altitudes of the Morley triangle. Thus, we have another theorem for isogonal axial $p\mathcal{K}$.

Theorem 5 *There are six isogonal axial $p\mathcal{K}$. The axes of symmetry are parallel to the sidelines of the Morley triangle.*

3.6 Oblique and axial $n\mathcal{K}$ isocubics

3.6.1 Oblique $n\mathcal{K}$

We shall now suppose that the isocubic is a $n\mathcal{K}$ with pole Ω, root P meeting $\mathcal{L}^\infty$ at F which is a flex on the cubic.

This isocubic has already four real common points A, B, C, F^* with $\mathcal{C}_\Omega$, the circumconic with perspector Ω, hence it must have two other common points whose isoconjugates are at infinity. This gives the following

Lemma 3 *Any isocubic $n\mathcal{K}$ passing through a real infinite point F contains the infinite points of $\mathcal{C}_\Omega$.*

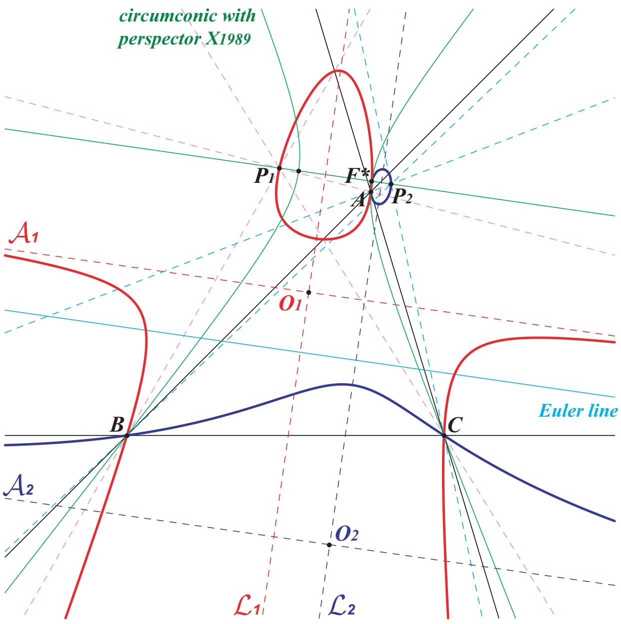

Figure 3.11: Two axial $p\mathcal{K}$

According to §1.5.1, the (real or not) asymptotes of the cubic at these two points pass through F^*.

We know that two isoconjugate points M and M^* on a $n\mathcal{K}$ share the same tangential. Since that of F is F, we deduce that the tangential of F^* must be F also, hence the tangent at F^* to the cubic is parallel to its real asymptote at F and the axis of symmetry must contain F^*.

Denote by U, V, W the traces of $\mathbb{P}(P)$ and by $N(P)$ the Newton line of the quadrilateral formed by $\mathbb{P}(P)$ and the sidelines of ABC. Following §1.5.3, the $n\mathcal{K}$ is the locus of M such that M and M^* are conjugated with respect to a fixed circle centered at the radical center of the circles with diameters AU, BV, CW and FF^* which turns out to be the infinte point of a perpendicular to $\mathbb{P}(P)$. Our circle degenerates into $\mathcal{L}^\infty$ and $\mathbb{P}(P)$.

On the other hand, the polar line of M in such degenerate circle is the homothetic of $\mathbb{P}(P)$ under $h_{M,2}$ therefore we have the following theorems.

Theorem 6 *Any $n\mathcal{K}$ with a flex at infinity is the locus of M such that $N(P)$ bisects MM^*. Furthermore, this flex must be the infinite point of $N(P)$ which means that P must lie on $N(F)$.*

Theorem 7 *The real asymptote $\mathcal{A}_F$ is the homothetic of $N(P)$ under $h_{F^*,2}$. It meets the cubic at X which lies on the circum-hyperbola γ_F passing through F, F^* and on the tangent at F^* to the cubic.*

Consequence

The polar conic $\mathcal{C}_F$ of F in the cubic is a hyperbola which passes through the midpoint of any two points M, N collinear with F. It obviously contains F and the midpoints of F^* and X^*, A and $UF^* \cap AF$, U and $AF^* \cap UF$, etc.

One of the asymptotes of $\mathcal{C}_F$ is XF and the other the parallel at F^* to the polar line of

F in $\mathcal{C}_\Omega$[6].

Let then P be a point on $N(F)$ and its trilinear polar $\mathbb{P}(P)$ meeting the sidelines of ABC at U, V, W. Any line passing through the infinite point F meets the cubic again at M, N. According to the theorem above, the midpoints of MM^* and NN^* lie on $N(P)$ hence M^*N^* contains F. Furthermore, MN^* and NM^* pass through F^*.

Conversely, any line L_F through F is the Newton line of some P lying on $N(F)$ which is easily constructed as follows : $h_{A,2}(L_F) \cap BC$, $h_{B,2}(L_F) \cap CA$, $h_{C,2}(L_F) \cap AB$ are three collinear points and P is the trilinear pole of the line passing through them.

Naturally, if we take P anywhere on $N(F)$, the corresponding $n\mathcal{K}(\Omega, P, F)$ is not an oblique (nor a central) cubic. We obtain the following theorems.

Theorem 8 *$n\mathcal{K}(\Omega, P, F)$ is an oblique cubic if and only if $X = F$ (at infinity) hence $X^* = F^*$.*

In this case, the real (inflexional) asymptote is that of γ_F passing through F (and the center O_F of γ_F). Thus, $N(P)$ is its homothetic under $h_{F^*,1/2}$. This easily gives the root P of the cubic on $N(F)$ and its construction follows with §1.5.4. The axis is the parallel at F^* to the polar line of F in $\mathcal{C}_\Omega$.

Theorem 9 *$n\mathcal{K}(\Omega, P, F)$ is an central cubic if and only if $X = F^*$ and $X^* = F$. The center is F^* and the real (inflexional) asymptote is FF^*.*

Refer back to §3.3 for another approach and more details.

3.6.2 Axial $n\mathcal{K}$

An oblique $n\mathcal{K}(\Omega, P, F)$ becomes an axial cubic if and only if its real asymptote is perpendicular to its axis which means that these two lines must be parallel to the axes of $\mathcal{C}_\Omega$. This gives two distinct situations :

1. $\mathcal{C}_\Omega$ is not a circle i.e. $\Omega \neq K$: there are only two axial non-isogonal $n\mathcal{K}$. The asymptote of one of them is parallel to the axis of the other.

   In particular, there are two isotomic axial $n\mathcal{K}$ with axes and asymptotes parallel to the axes of the Steiner ellipse. The figure 3.12 shows one of the two cubics.

2. $\mathcal{C}_\Omega$ is a circle (the circumcircle) i.e. $\Omega = K$: there are infinitely many axial isogonal $n\mathcal{K}$. They are all focal cubics forming a net of cubics with singular focus on the circumcircle. See §4.1.4.

3.7 $p\mathcal{K}$ and harmonic homologies

3.7.1 Introduction

In paragraphs 3.1, 3.2, 3.5 we have met central, oblique and axial isocubics $p\mathcal{K}$. Since these transformations are special cases of harmonic homologies, we now generalize and characterize isocubics which are invariant under such homologies.

Let us denote by h the harmonic homology with center $X = \alpha : \beta : \gamma$ and axis $\mathcal{L}$, the trilinear polar of the point $Q = l : m : n$. We suppose that X does not lie on $\mathcal{L}$.

[6] In other words, this second asymptote is parallel to the conjugated diameter of the real asymptote $\mathcal{A}_F$ with respect to $\mathcal{C}_\Omega$. It is the locus of the center of γ_F when P traverses $N(F)$.

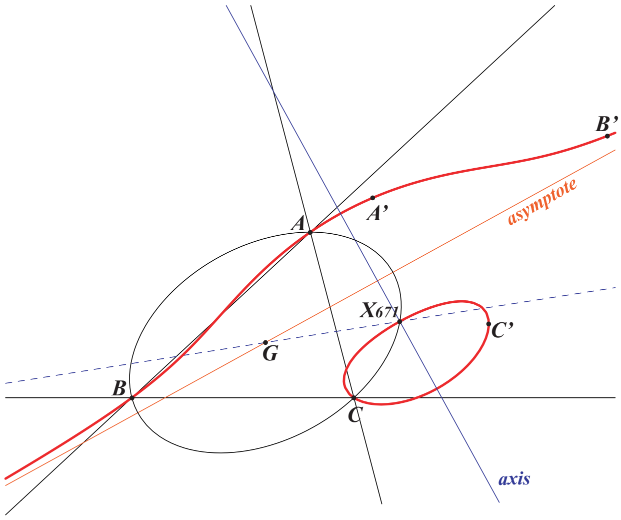

Figure 3.12: An axial isotomic $n\mathcal{K}$

Remember that h maps any point $M = x : y : z$ (distinct of X) to the point M' harmonic conjugate of M with respect to X and $\mathcal{L} \cap XM$. The barycentric coordinates of M' are

$$(-mn\alpha + nl\beta + lm\gamma)x - 2l\alpha(ny + mz) : :$$

or

$$\left(-\frac{\alpha}{l} + \frac{\beta}{m} + \frac{\gamma}{n}\right) x - 2\alpha \left(\frac{y}{m} + \frac{z}{n}\right) : : \; .$$

When $\mathcal{L} = \mathcal{L}^\infty$, h is a central symmetry. When X is an infinite point, h is an oblique or axial symmetry.

3.7.2 Theorem

For a given harmonic homology h, there is a unique cubic $p\mathcal{K} = p\mathcal{K}(\Omega, P)$ with pole Ω, pivot P which is invariant under h.

Its pole Ω is the barycentric product of Q and Q/X with coordinates

$$p : q : r = l\alpha \left(-\frac{\alpha}{l} + \frac{\beta}{m} + \frac{\gamma}{n}\right) : : \; .$$

In other words, the cubic is invariant in the isoconjugation which swaps Q and the cevian quotient Q/X but these two points do not necesseraly lie on the cubic.

Its pivot is $h(X^*)$ where

$$X^* = l \left(-\frac{\alpha}{l} + \frac{\beta}{m} + \frac{\gamma}{n}\right) : : ,$$

hence

$$P = u : v : w = l \left(-3\,\frac{\alpha}{l} + \frac{\beta}{m} + \frac{\gamma}{n}\right) : : \; .$$

Note that Q, X, X^* and P/X, Q/X, X^* are collinear.

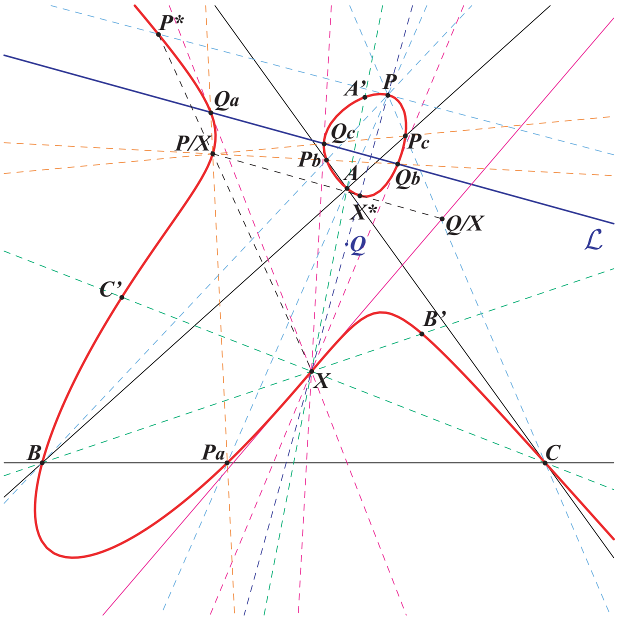

Figure 3.13: $p\mathcal{K}$ invariant under harmonic homology

3.7.3 Properties

1. X is a flex on the cubic. Its polar conic decomposes into the harmonic polar $\mathcal{L}$ and the inflexional tangent at X. This tangent passes through Q/X. See figure 3.13.

2. The cubic contains $A' = h(A)$, $B' = h(B)$, $C' = h(C)$ where

$$A' = -\frac{\alpha}{l} + \frac{\beta}{m} + \frac{\gamma}{n} : -2\,\frac{\beta}{l} : -2\,\frac{\gamma}{l}.$$

3. The cubic meets $\mathcal{L}$ at three always real (and rational) points Q_a, Q_b, Q_c and the tangents to the cubic at these points concur at X. We have

$$Q_a = 2\alpha : \beta - m\left(\frac{\alpha}{l} + \frac{\gamma}{n}\right) : \gamma - n\left(\frac{\alpha}{l} + \frac{\beta}{m}\right),$$

   the other points similarly. Obviously, these three points are the only points of the cubic fixed by the homology. Furthermore, Q_a is the intersection of the lines BC' and CB'.

4. The three triangles ABC, $A'B'C'$, $Q_aQ_bQ_c$ (although degenerate) are triply perspective at X, X^* and P hence this type of cubic can be seen as a Grassmann cubic.

   The cevian triangle $P_aP_bP_c$ of P is also perspective with $Q_aQ_bQ_c$ at P/X, another point on the cubic and on the line XP^* (see §1.4.1, remark 2).

5. For any point M on the cubic, we find the following collinearities :

   – M, X, $h(M)$,

   – M, M^*, P,

   – M, P^*, P/M,

– M, P/X, $h(M^*)^*$.

This easily gives the polar conic of M in the cubic.

6. If M and M^* are two isoconjugate points on the cubic then X^*, $h(M)$, $h(M^*)$ are three collinear points on the cubic.

7. For any M on the cubic, the polar conics of M and $h(M)$ meet at four points which are the poles of the line $Mh(M)$ in the cubic. Since $Mh(M)$ passes through X, these poles lie two by two on $\mathcal{L}$ and on the inflexional tangent at X.

8. If X' and X'' are the two other real flexes on the cubic collinear with X, the cubic is also invariant under two other homologies with centers at these points. The three corresponding axes $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{L}''$ are concurrent.

3.7.4 $p\mathcal{K}$ with given axis and pole

Theorem : for a given axis $\mathcal{L}$ and a given pole Ω, there is a unique $p\mathcal{K}$ invariant under a harmonic homology.

The center of the harmonic homology is

$$X = \frac{p}{l}\left(-\frac{p}{l^2} + \frac{q}{m^2} + \frac{r}{n^2}\right) : : ,$$

the isoconjugate of the trilinear pole Q of $\mathcal{L}$ in the isoconjugation with pole Q^2/Ω.

Examples :

- with $\Omega = K$ and $\mathcal{L} = \mathcal{L}^\infty$, we obtain the Darboux cubic, actually a central cubic centered at $X = O$.

- with $\Omega = K$ and $\mathcal{L} = GK$, we obtain the cubic with center X_{39}, pivot $P = a^4 + b^2c^2 + 3a^2(b^2 + c^2) : :$.

- with $\Omega = G$ and $\mathcal{L} = \mathbb{P}(X_{75})$, we obtain the cubic with center X_{63}, pivot $P = (b^2 - c^2)^2/a + a(2b^2 + 2c^2 - 3a^2) : :$. (isogonal conjugate of X_{2155}) passing through G, X_{92}, X_{189}, X_{329}, X_{2184}. The inflexional tangent at X_{63} passes through I. This cubic is the isogonal transform of $p\mathcal{K}(X_{32}, X_{610})$. See figure 3.14.

3.8 $n\mathcal{K}$ and harmonic homologies

This paragraph borrows several ideas by Wilson Stothers.

3.8.1 Generalities

Let us consider the cubic $\mathcal{K} = n\mathcal{K}(\Omega, P, Q)$ with pole $\Omega = p : q : r$, root $P = u : v : w$ passing through $Q = \alpha : \beta : \gamma$ (not lying on a sideline of ABC) and obviously through its isoconjugate Q^*.

This cubic has equation :

$$\alpha\beta\gamma \sum_{\text{cyclic}} ux(ry^2 + qz^2) = xyz \sum_{\text{cyclic}} u\alpha(r\beta^2 + q\gamma^2).$$

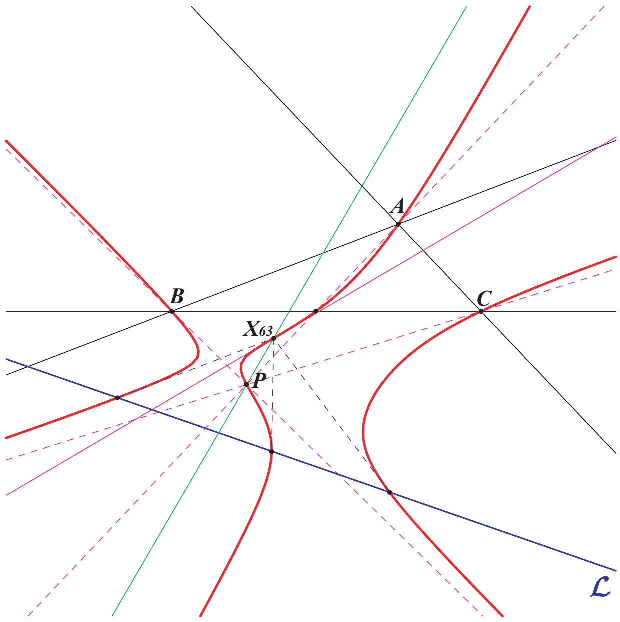

Figure 3.14: Isotomic $p\mathcal{K}$ invariant under harmonic homology

Q is a flex or a node on the cubic if and only if it lies on the hessian of the cubic which expresses that the polar conic $\mathcal{C}(Q)$ of Q is degenerate. A calculation shows that this condition is equivalent to

$$\sum_{\text{cyclic}} \alpha v w = 0 \tag{3.1}$$

or

$$\sum_{\text{cyclic}} p\alpha u (r\beta^2 - q\gamma^2)^2 = 0. \tag{3.2}$$

The equation (3.1) shows that Q must lie on $\mathbb{P}(P)$ i.e. P must lie on $\mathcal{C}(Q)$, the circum-conic with perspector Q. But $\mathcal{K}$ already meets $\mathbb{P}(P)$ at three points (on the sidelines of ABC) hence it must degenerate into $\mathbb{P}(P)$ and the circum-conic with perspector P^* which is its isoconjugate.

The equation (3.2) can be construed as

1. Q must lie on the circum-quintic with equation

$$\sum_{\text{cyclic}} pux(ry^2 - qz^2)^2 = 0 \tag{3.3}$$

passing through the fixed points of the isoconjugation in which cases $\mathcal{K}$ is a $c\mathcal{K}$.

2. Ω must lie on the $c\mathcal{K}$ with equation

$$\sum_{\text{cyclic}} \alpha ux(z\beta^2 - y\gamma^2)^2 = 0 \tag{3.4}$$

with singularity the barycentric square Q^2 of Q, with root the barycentric product $Q \times R$ of Q and R.

3. P must lie on the line $\mathcal{L}(\Omega, Q)$ with equation

$$\sum_{\text{cyclic}} p\alpha(r\beta^2 - q\gamma^2)^2\, x = 0, \tag{3.5}$$

which is $\mathbb{P}(P_1)$ where $P_1 = \dfrac{1}{p\alpha(r\beta^2 - q\gamma^2)^2} : : .$

This line $\mathcal{L}(\Omega, Q)$ contains the two following points :

• $P_2 = \dfrac{1}{\alpha(r\beta^2 - q\gamma^2)} : :$, the trilinear pole of the line QQ^*,

• $P_3 = \dfrac{\alpha}{p(r\beta^2 - q\gamma^2)} : :$, the Q^2-isoconjugate of $\mathbb{P}(Q) \cap \mathbb{P}(Q^*)$.

Note that these two points lie on $\mathcal{C}(Q)$.

In both cases, the cubic degenerates :

• $n\mathcal{K}(\Omega, P_2, Q)$ is the union of the line QQ^* and the circum-conic which is its isoconjugate,

• $n\mathcal{K}(\Omega, P_3, Q)$ is the union of $\mathbb{P}(P_3)$ (which contains Q) and the circum-conic which is its isoconjugate with perspector P_3^*.

3.8.2 $n\mathcal{K}(\Omega, P, Q)$ with P on $\mathcal{L}(\Omega, Q)$

Let then P be a point on $\mathcal{L}(\Omega, Q)$ distinct of P_2 and P_3.

All the corresponding cubics $n\mathcal{K}(\Omega, P, Q)$ form a pencil of cubics which is generated by the two decomposed cubics $n\mathcal{K}(\Omega, P_2, Q)$ and $n\mathcal{K}(\Omega, P_3, Q)$.

For any cubic $\mathcal{K}$ of the pencil, we have the following properties :

1. $\mathcal{K}$ passes through A, B, C, Q, Q^*,

2. the tangent at Q is the line $\mathbb{P}(P_3)$,

3. the tangent at Q^* is the line QQ^*,

   Note that these two lines are independent of P and intersect at Q.

4. The polar conic of Q is the union of $\mathbb{P}(P_3)$ and the harmonic polar $\mathcal{H}(\Omega, Q)$ of Q in $\mathcal{K}$. $\mathcal{H}(\Omega, Q)$ passes through Q^* and, for any point M on this line, the polar line of M in $\mathcal{K}$ passes through Q.

   Thus, $\mathcal{K}$ must meet $\mathcal{H}(\Omega, Q)$ at two other (real or not) points Q_1, Q_2 which are the common points of $\mathcal{H}(\Omega, Q)$ and the conic which is its isoconjugate (this conic contains Q). Obviously, the tangents at Q^*, Q_1, Q_2 to the cubic concur at Q.

   It follows that any line through Q meets the cubic $\mathcal{K}$ at two other points M, M' and the line $\mathcal{H}(\Omega, Q)$ at N such that N is the harmonic conjugate of Q with respect to M, M'. In other words, $\mathcal{K}$ is invariant under the harmonic homology with axis $\mathcal{H}(\Omega, Q)$ and center Q.

   Several special cases are interesting :

   • when Q lies at infinity, $\mathcal{K}$ is an oblique $n\mathcal{K}$ (see §3.6.1),

   • when Q^* lies at infinity, $\mathcal{K}$ is a central $n\mathcal{K}$ (see §3.3),

3.8.3 Special $n\mathcal{K}(\Omega, P, Q)$ with P on $\mathcal{L}(\Omega, Q)$

We have already seen that we obtain two decomposed cubics when P is either P_2 or P_3.

When P is choosen so that $n\mathcal{K}(\Omega, P, Q)$ contains one of the fixed points of the iso-conjugation, the cubic is a $c\mathcal{K}$.

In general, this pencil of cubics also contains one and only one $n\mathcal{K}_0$ when P is the point

$$P_4 = p(r\beta^2 - q\gamma^2)[p(p\beta^2\gamma^2 + q\gamma^2\alpha^2 + r\alpha^2\beta^2) - 3qr\alpha^4]/\alpha : : ,$$

the common point of $\mathcal{L}(\Omega, Q)$ and the line $\mathcal{D}(\Omega, Q)$ with equation

$$\sum_{\text{cyclic}} \alpha(r\beta^2 + q\gamma^2)\, x = 0. \tag{3.6}$$

$\mathcal{D}(\Omega, Q)$ is $\mathbb{P}(Z^*)$ where $Z = p\alpha(r\beta^2 + q\gamma^2) : :$, the pole of the line QQ^* in the circum-conic through Q and Q^*.

Naturally, if these two lines are equal, all the cubics of the pencil are $n\mathcal{K}_0$. This arises when

$$\sum_{\text{cyclic}} qr\alpha^4[\alpha(r\beta - q\gamma)(r\beta^2 + q\gamma^2) - 3p\beta\gamma(r\beta^2 - q\gamma^2)] = 0 \tag{3.7}$$

which can be construed as Ω lies on a circum-quartic $\mathcal{Q}(Q)$.

For example, with $Q = K$, we find that the quartic contains X_{184}, the barycentric product of K and O, and that P must lie on the line $X_{110}X_{112}$ containing the point $E_{409} = a^2(b^2 - c^2)S_A(b^4 + c^4 - a^4 - b^2c^2) : :$. This gives the cubic $n\mathcal{K}_0(X_{184}, E_{409})$ with center K, with axis OX_{647}, with inflexional tangent KX_{25}. See figure 3.15.

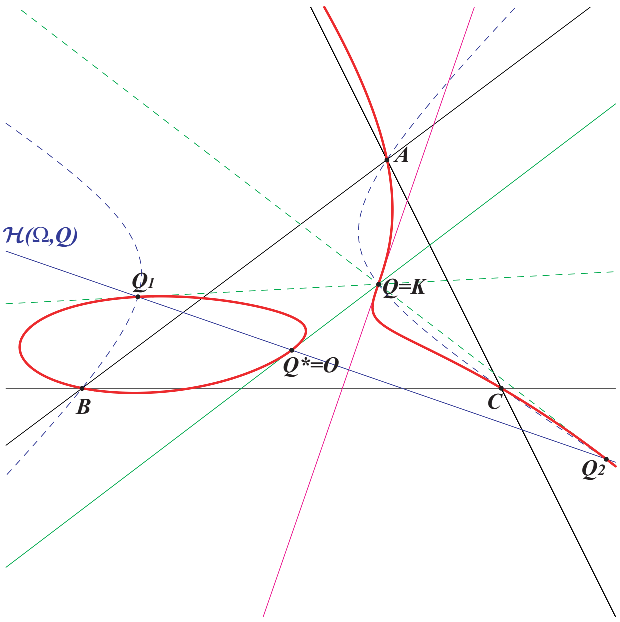

Figure 3.15: $n\mathcal{K}_0$ invariant under harmonic homology

Chapter 4

Circular isocubics
Inversible isocubics

This chapter is a generalization of [20]. See also [10, 11]. It ends with inversible isocubics which are a special case of circular isocubics. A sequel of this chapter is found in §8.5.

Some "unusual" examples of such cubics are shown.

Since the two circular points at infinity are isogonal conjugates, we have two different situations examined in the two following paragraphs.

4.1 Main theorems for isogonal cubics

See [14] for details.

4.1.1 Pivotal isogonal circular cubics

- **Theorem : all pivotal isogonal circular cubics are in the same pencil of cubics.**

  An isogonal $p\mathcal{K}$ is circular if and only if its pivot P is on $\mathcal{L}^{\infty}$. All the cubics pass through A, B, C, the four in/excenters and the two circular points at infinity.

  $\mathbf{g}P$ is a point on the circumcircle : it is the intersection of the curve with its real asymptote. The singular focus F is the antipode of this point. The line OP is the orthic line of the cubic : the polar conic of any of its points is a rectangular hyperbola.

  The most famous is the Neuberg cubic **K001** with pivot X_{30} and singular focus X_{110}. It is the only cubic of this type that contains O (and H).

  Another worth noticing is **K269**, the one with pivot X_{515} (point at infinity of IH). Its singular focus is X_{109} and it also passes through X_{36}, X_{40}, X_{80}, X_{84}, X_{102}.

  See [9] for other examples.

- **Pivotal isogonal focal cubics**

  It is easy to see that there are only three pivotal isogonal focal circular cubics (i.e. Van Rees focals) : they are obtained when the pivot is the point at infinity of one of the altitudes of ABC.

Indeed, the isogonal conjugate of the point at infinity of one of the altitudes of ABC is the antipode of the corresponding vertex of ABC on the circumcircle. Thus, the singular focus of such focal cubic must be a vertex of ABC and the focal tangent contains O.

More precisely, let $p\mathcal{K}_A$ be the cubic whose pivot is the infinite point of the altitude AH. We have the following properties, see figure 4.1.

1. the singular focus of $p\mathcal{K}_A$ is A and its polar conic is the $A-$Apollonian circle of ABC. This is the circle with diameter the intersections of BC with the two bisectors of angle A. This circle contains A and the isodynamic points X_{15}, X_{16}.
2. the focal tangent at A passes through O and the antipode A_O of A on the circumcircle. A_O is the tangential of A on $p\mathcal{K}_A$ and the isopivot of $p\mathcal{K}_A$.
3. the real asymptote is the parallel at A_O to the altitude AH or equivalently, the reflection about O of this altitude.
4. the orthic line of the cubic is the perpendicular bisector Δ_A of BC hence the polar conic of any point on Δ_A is a rectangular hyperbola whose center lies on the circle E_A which is the reflection of the $A-$Apollonian circle about Δ_A. This circle E_A is in a way the Euler circle of the cubic.
5. in particular, the polar conic of the infinite point of Δ_A is the diagonal rectangular hyperbola whose center is the reflection A' of A about Δ_A. Note that this point A' lies on the circumcircle. This diagonal rectangular hyperbola contains the in/excenters of ABC and the midpoint of BC.
6. it follows that any circle with center Ω on Δ_A which is orthogonal to E_A (and therefore passing through B and C) meets the line $A\Omega$ at two points on $p\mathcal{K}_A$.
7. $p\mathcal{K}_A$ is an isogonal $n\mathcal{K}$ with respect to infinitely many triangles. Let M_1, M_2 be two points on $p\mathcal{K}_A$. The circum-circle of AM_1M_2 meets $p\mathcal{K}_A$ again at M_3 which is the reflection about the diameter passing through A of the isogonal conjugate (with respect to the triangle AM_1M_2) of the infinite point of the altitude AH.

   Now, if M'_i is the reflection of M_i in Δ_A, the parallel at M'_i to Δ_A meets the line M_jM_k at U_i on the cubic. These three points U_i are collinear and the trilinear pole of the line $U_1U_2U_3$ with respect to the triangle $M_1M_2M_3$ is the root of the cubic.

   For any point Z on $p\mathcal{K}_A$, the isogonal conjugates $\mathbf{g}Z$, Z^* of Z in ABC, $M_1M_2M_3$ respectively lie on a line passing through A. Since Z^* also lies on the reflection of the line $Z\mathbf{g}Z$ in Δ_A (hence parallel to Δ_A), this point Z^* is independent of the triangle $M_1M_2M_3$.

   This property can actually be adapted to any focal cubic.
8. $p\mathcal{K}_A$ has three real prehessians and one of them $p\mathcal{H}_A$ is a stelloid with radial center A. Its asymptotes are the parallels at A to those of the McCay cubic. Obviously, the nine common points of $p\mathcal{K}_A$ and $p\mathcal{H}_A$ are the points of inflexion of the two curves.
9. apart isogonal conjugation, $p\mathcal{K}_A$ is invariant under seven other transformations namely :
   - four inversions, each one with pole one in/excenter which swaps one vertex of ABC and the corresponding other collinear in/excenter. These in/excenters are called centers of anallagmaty of the focal cubic.

– three involutions swapping any point of the plane with the center of its polar conic with respect to one of the three prehessians. When we take $p\mathcal{H}_A$, this involution is given by :

$$(x : y : z) \mapsto (a^2yz + b^2zx + c^2xy : -b^2z(x + y + z) : -c^2y(x + y + z)).$$

This is the commutative product of a reflection about a $A-$bisector of triangle ABC and an inversion in the circle with center A, radius $\sqrt{bc}$. The two other involutions are similar and the two corresponding circles have their centers on the $A-$median of ABC.

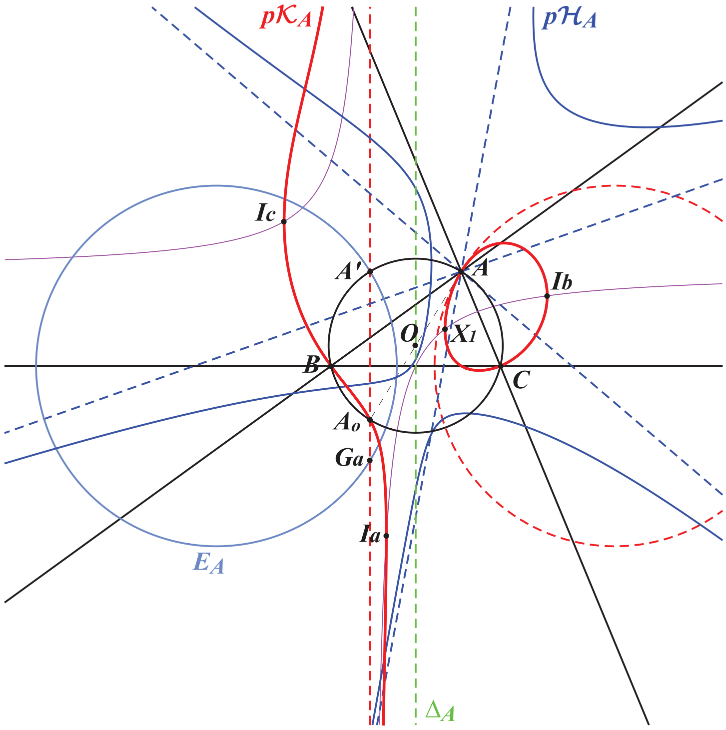

Figure 4.1: The pivotal isogonal focal cubic $p\mathcal{K}_A$

- **Pivotal isogonal degenerate cubics**

  A pivotal isogonal cubic degenerates into a line and a circle if and only if P is the point at infinity of one of the six bisectors. In this case, the line is the bisector and the circle passes through the remaining in/excenters and vertices of ABC.

- **Construction of a pivotal isogonal circular cubic knowing its pivot**

  Let ℓ be the line through I (incenter) and P (at infinity) and let (c) be a variable circle tangent at I to ℓ, centered at ω on the perpendicular at I to ℓ.

  The perpendicular at $\mathbf{g}P$ to the line ωF meets (c) at M and N on the cubic. Following §1.4, remark 2, we see that $N = \mathbf{g}\mathrm{P}/M$

- **Further remarks**

  1. The real asymptote is parallel at $\mathbf{g}P$ to ℓ and meets the circumcircle again at E.

When P traverses $\mathcal{L}^\infty$, the real asymptote envelopes a deltoid, the contact point being the symmetric of E about $\mathbf{g}P$. This deltoid is the anticomplement of the Steiner deltoid $\mathcal{H}_3$. It is the envelope of axis of inscribed parabolas in triangle ABC.

2. The polar conic of P is the rectangular hyperbola centered at E passing through the four in/excenters. ABC is autopolar in this hyperbola.

   There are two other points P_1 and P_2 on the cubic (not always real) whose polar conics are rectangular hyperbolas : they lie on the line OP and, since they are isogonal conjugate, on the rectangular hyperbola $ABCH\mathbf{g}P$ as well. From this it is obvious they are two points on the McCay cubic and their midpoint is on the polar conic of P.

3. The line $\mathbf{g}M\mathbf{g}N$ passes through a fixed point $Q = P/\mathbf{g}P$ of the cubic which is the common tangential of $\mathbf{g}P$ and the vertices of the cevian triangle of P.

4. The tangents at M and $\mathbf{g}M$ to the cubic pass through the isogonal conjugate of the midpoint of $M\mathbf{g}M$.

5. The poloconic of $\mathcal{L}^\infty$ is the hyperbola with focus F, directrix the parallel at O to the Simson line of F, eccentricity 2. One of its vertices is E and the other S is the homothetic of E under $h(F, 1/3)$. Note that the line FE is the focal axis and that the asymptotes are obtained by rotating this focal axis about the center of the hyperbola with angles $+60°$ and $-60°$.

   For any choice of P, this poloconic passes through the vertices of the circumtangential triangle formed by the points T_i on the circumcircle such that the line $T_i\,\mathbf{g}T_i$ is a tangent to the circumcircle. See figure 4.2 for an illustration with the Neuberg cubic **K001**.

6. The in/excenters are four centers of anallagmaty of the cubic. In other words, the cubic is invariant under four inversions with pole one in/excenter I_x, each swapping one vertex of ABC and the in/excenter that lies on the line through this vertex and I_x.

4.1.2 Non-pivotal isogonal circular cubics

- **Theorem : all non-pivotal isogonal circular cubics are in the same net of cubics.**

  They all are Van Rees focals with singular focus on the circumcircle.

  A computation shows that there is only one non-pivotal isogonal circular cubic with given root $P(u : v : w)$. Its equation is :

$$2(S_A u + S_B v + S_C w)xyz + \sum_{\text{cyclic}} ux(c^2y^2 + b^2z^2) = 0$$

or

$$\sum_{\text{cyclic}} ux(c^2y^2 + b^2z^2 + 2S_A\,yz) = 0$$

in which we recognize the equation of the circle centered at A passing through O. Hence, the net is generated by three decomposed cubics, one of them being the union of this circle and the sideline BC, the two others similarly.

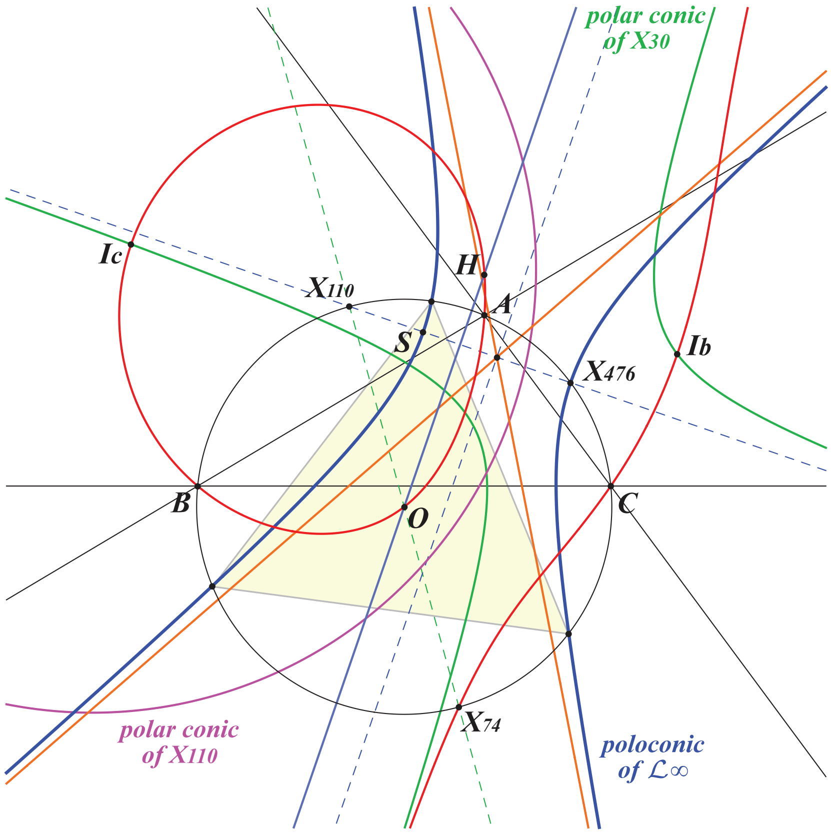

Figure 4.2: Polar conics and Poloconic of $\mathcal{L}^\infty$ in an isogonal $p\mathcal{K}$

The third (real) point on $\mathcal{L}^\infty$ is $Z = (v - w : w - u : u - v)$ which is the point at infinity of $\mathbb{P}(\mathbf{t}P)$ with equation $ux + vy + wz = 0$.

The real asymptote is obviously parallel to this line.

The singular focus is $\mathbf{g}Z = [a^2/(v - w) : b^2/(w - u) : c^2/(u - v)]$ clearly on the circumcircle and on the cubic.

This cubic is the locus of foci of inscribed conics in the quadrilateral formed by the four lines AB, BC, CA and $UVW = \mathbb{P}(P)$. The singular focus F of the cubic is the Miquel point of the quadrilateral.

Equivalently, this cubic is the locus of foci of inscribed conics

– with center on the Newton line of this quadrilateral which is $\mathbb{P}(\mathbf{ta}P)$. Note that the Monge (orthoptic) circles of these conics form a pencil of circles passing through the (not always real) antiorthocorrespondents of P (see [31] and [22]).

– with perspector on the circum-conic with perspector P.

- Two points R and S that are not isogonal conjugates define one and only one cubic $n\mathcal{K}$ of the net denoted by $n\mathcal{K}_{RS}$. Then $n\mathcal{K}_{RS}$ passes through $\mathbf{g}R$, $\mathbf{g}S$ and through $R_1 = RS \cap \mathbf{g}R\mathbf{g}S$, $R_2 = R\mathbf{g}S \cap S\mathbf{g}R$, R_3 orthogonal projection on R_1R_2 of $R\mathbf{g}R \cap S\mathbf{g}S$. The singular focus F (on the circumcircle) is the second intersection of the circles R_2RS and $R_2\mathbf{g}R\mathbf{g}S$.

  If Δ is the line through the midpoints of $R\mathbf{g}R$ and $S\mathbf{g}S$ [1], $n\mathcal{K}_{RS}$ is the locus of

[1] These midpoints are distinct since R and S are not isogonal conjugates.

foci of inscribed conics whose center is on Δ and the real asymptote of $n\mathcal{K}_{RS}$ is $h_{F,2}(\Delta)$ [2].

Let us call now α, β, γ the points where Δ meet the sidelines of the medial triangle of ABC and U, V, W the symmetrics of A, B, C about α, β, γ respectively. U, V, W are collinear and the trilinear pole of this line is the root P of the cubic.

Now let R' be the intersection of the line $F\mathbf{g}R$ with the parallel at R to the asymptote and define S', $\mathbf{g}R$', $\mathbf{g}S$' likewise. The four circles with diameters $R\mathbf{g}R$', $\mathbf{g}RR'$, $S\mathbf{g}S$', $\mathbf{g}SS'$ are in the same pencil $\mathcal{F}$ whose axis is Δ and whose radical axis is denoted by Δ'. The parallel at F to the asymptote intersects Δ' at $\mathbf{g}X$ whose isogonal conjugate is the point X where $n\mathcal{K}_{RS}$ meets its asymptote.

From all this, we see that $n\mathcal{K}_{RS}$ is :

- the locus of intersections of circle of $\mathcal{F}$ centered at ω with the line $F\omega$.
- the locus of intersections of circle (of the pencil $\mathcal{F}'$) centered at Ω on Δ' and orthogonal to $\mathcal{F}$ with the perpendicular at X to $F\Omega$.
- the locus of point M from which the segments RS and $\mathbf{g}R\mathbf{g}S$ (assuming they are not equipollent) are seen under equal or supplementary angles. For this reason, this cubic is called an isoptic cubic.

Remark :

The pencils of circles $\mathcal{F}$ and $\mathcal{F}'$ are orthogonal : the base points of one are the Poncelet points of the other. Those two points are on $n\mathcal{K}_{RS}$ and are the contacts of the two tangents from F.

- The cubic has a singularity if and only if the quadrilateral is circumscribed to a circle i.e. if and only if UVW is tangent to one of the in/excircles i.e. if and only if the root P is on one of the trilinear polars of the Gergonne point or one of its harmonic associates. (See §4.3.2 below)

  The singularity is the center of the circle and, consequently, those cubics are the only conico-pivotal [3] cubics we can find among non-pivotal isogonal circular cubics. Then, the pivot (remember it is a conic) is the envelope of the focal axes of inscribed conics in the quadrilateral and therefore a parabola tangent to the three diagonals of the complete quadrilateral.

- **Remark :**

  When P lies on the orthic axis, the term in xyz vanishes and we get a pencil of $n\mathcal{K}_0$ with a simpler equation :

$$\sum_{\text{cyclic}} ux(c^2y^2 + b^2z^2) = 0 \quad \text{with} \quad S_Au + S_Bv + S_Cw = 0.$$

  All these cubics contain the four foci of the inscribed conic with center the Lemoine point K.

  See several examples in §4.3.3 and §4.3.4 below.

[2] This asymptote is perpendicular to the Simson line of F.

[3] See [24] and §7.3.3, §8 below

4.1.3 Non-pivotal isogonal circular cubic with given root

We provide a detailed construction of such cubic assuming we know its root $P \neq G$.[4] We suppose that the cubic is not unicursal. See Chapter 8 to cover this case.

1. Draw $\Delta_P = \mathbb{P}(P)$ intersecting ABC sidelines at U, V, W and the Newton line Δ of the complete quadrilateral (passing through the midpoints of AU, BV, CW).

2. The isogonal conjugate of the point at infinity of Δ is F (singular focus) and the homothetic of Δ (center F, ratio 2) is the real asymptote.

3. Let A' be the intersection of AF and the parallel at U to Δ (B', C' likewise) and A'' the intersection of UF and the parallel at A to Δ (B'', C'' likewise). These six points lie on the cubic.

   Let us call $\mathcal{F}$ the pencil of circles containing the circles with diameters AA', BB', CC', UA'', VB'', WC''. Its axis is Δ and its radical axis is Δ' intersecting the parallel at F to Δ at X^* isogonal conjugate of X, intersection of the cubic with its asymptote. The tangent at F to the cubic is FX.

4. Each circle γ of $\mathcal{F}$ centered at ω on Δ intersects the line $F\omega$ at two points M and N which are on the cubic and their isogonal conjugates $\mathbf{g}M$, $\mathbf{g}N$ as well. Notice that $M\mathbf{g}M$ and $N\mathbf{g}N$ meet on Δ' and that $M\mathbf{g}N$ and $\mathbf{g}MN$ are parallel to the asymptote. The base-points (real or not) of $\mathcal{F}$ are on the cubic.

5. The conic through the midpoints of AA'', BB'', CC'', UA', VB', WC', $F\mathbf{g}X$ is the polar conic of the point at infinity of the asymptote. This is a rectangular hyperbola centered at Ω on the asymptote and on the circle Γ with diameter FX belonging to $\mathcal{F}$. The intersection of Δ and Δ' lies on the rectangular hyperbola whose intersections with Γ are two points on the cubic.

6. the polar conic of F is the circle through F centered at the intersection of Δ' and the perpendicular at F to FX.

7. the circle Γ meets Δ at E_1 and E_2. The lines FE_1 and FE_2 are the bisectors at F of the lines FX, FX^*. These lines intersect the rectangular hyperbola above at four (real or not) points which are the centers of anallagmaty O_i of the cubic. In other words, the cubic is invariant under the inversions with pole O_i swapping F and the other point O_j collinear with F and O_i.

   More precisely,

   – if the base-points of $\mathcal{F}$ are real then the cubic is bipartite and the four centers of anallagmaty O_i are all real,

   – if the base-points of $\mathcal{F}$ are not real then the cubic is unipartite and only two of the four centers of anallagmaty O_i are real.

4.1.4 Non-pivotal isogonal circular cubics with given focus

Let F be a given point on the circumcircle and $\mathbf{g}F$ its isogonal conjugate at infinity. All the non-pivotal isogonal circular (focal) cubics with focus F form a pencil of cubics and their roots lie on $\mathbb{P}(\mathbf{tg}\mathrm{F})= \mathcal{L}_F$. This line always contains G since $\mathbf{tg}F$ is a point on the Steiner ellipse collinear with F and the Steiner point X_{99}. Notice that the trilinear

[4] When $P = G$, the cubic decomposes into $\mathcal{L}^{\infty}$ and the circumcircle.

polars of the roots are tangent to the inscribed parabola with focus F, directrix the Steiner line of F and perspector the point $\mathbf{tg}F$. All the cubics have obviously parallel real asymptotes and each one meets its asymptote at a point X lying on the circum-conic Γ_F through F and $\mathbf{g}F$, the isogonal transform of the line $F\mathbf{g}F$. The tangent at F to Γ_F meets the circumcircle again at Φ and the perpendicular at Φ to this tangent is the locus of the centers of the circles which are the polar conics of F.

Let then Ω_F be a point on this latter line and γ_F the circle with center Ω_F passing through F. The tangent T_F at F to γ_F meets Γ_F at the point X. The parallel at the midpoint of FX to $F\mathbf{g}F$ is the Newton line of a quadrilateral such as the sought cubic is the locus of foci of inscribed conics in this quadrilateral. The homothetic of this Newton line under $h(A, 2)$ meets the line BC at U and V, W are defined similarly. These three points are collinear and the trilinear pole of the line is the root P of the cubic.

To realize the construction, take a point on the perpendicular at Ω_F to $F\mathbf{g}F$ and draw the circle centered at this point orthogonal to the circle with diameter FX. This circle belongs to the pencil of circles containing γ_F and having the Newton line as radical axis with it. The contacts M, N of the tangents drawn through F to this circle are two points on the cubic.

Remark :

- The isogonal conjugate of X is the intersection of $F\mathbf{g}F$ with its perpendicular at Ω_F .
- The following triads of points are collinear : X, M, N - X, $\mathbf{g}M$, $\mathbf{g}N$ - $\mathbf{g}X$, M, $\mathbf{g}N$ - $\mathbf{g}X$, N, $\mathbf{g}M$.
- The perpendicular at Ω_F to $F\mathbf{g}F$ meets the circle with diameter FX at two (real or not) isogonal conjugate points and the tangents at these points pass through X. These two points also lie on the circle orthogonal to γ_F whose center is the intersection of the Newton line and the perpendicular at F to XF. This circle and this perpendicular meet at the two centers of anallagmaty E_1, E_2 of the cubic.
- The circle γ_F meets the Newton line at two (real or not) points on the cubic and the tangents at these points pass through F. These two points lie on the isogonal circular $p\mathcal{K}$ with pivot $\mathbf{g}F$. Notice that the reflection F' of F in Φ is the tangential of F in this $p\mathcal{K}$.

Special cases :

The aforementioned pencil contains :

- one degenerated cubic into the circumcircle and $\mathcal{L}^\infty$ obtained when $P = G$.
- one $n\mathcal{K}_0$ obtained when P is the intersection of the orthic axis and $\mathcal{L}_F$.
- four strophoids with nodes the in/excenters obtained when P is the intersection of the trilinear polar of the Gergonne point (or an extraversion) and $\mathcal{L}_F$. This happens when γ_F contains one of these in/excenters.
- one central cubic obtained when Ω_F is the infinite point of the perpendicular at Φ to $F\Phi$.

- one "axial" cubic obtained when Ω_F lies on the perpendicular at F to $F\mathbf{g}F$, the axis of symmetry being this perpendicular. The real asymptote is the asymptote of Γ_F which is parallel to the line $F\mathbf{g}F$.

The figure 4.3 shows the pencil of cubics obtained when $F = X_{99}$.

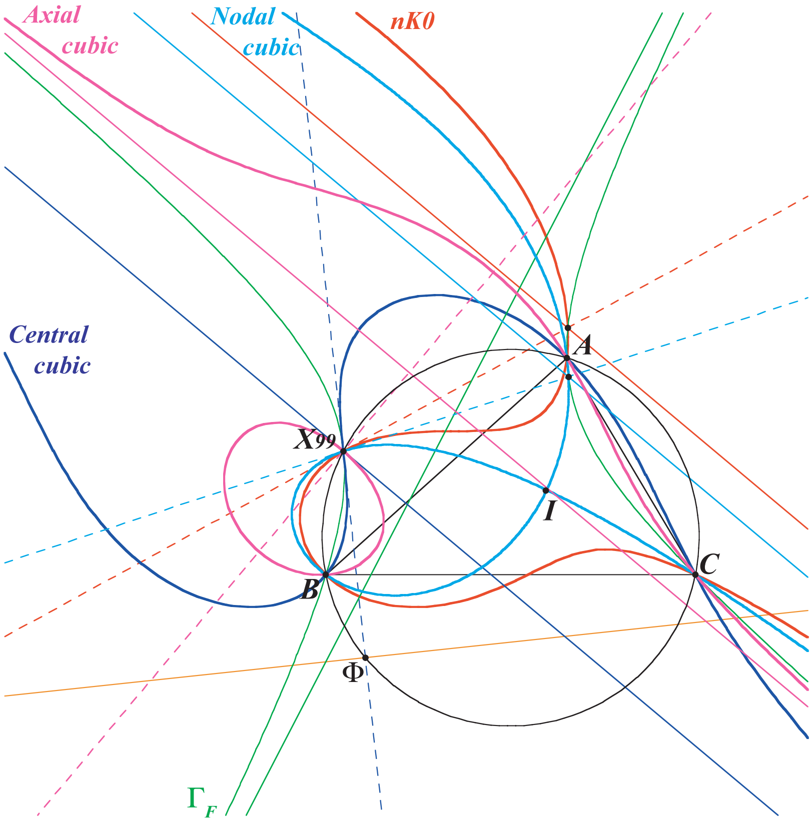

Figure 4.3: Pencil of focal cubics with focus X_{99}

4.2 Main theorems for non-isogonal cubics

Throughout this section, we take $\Omega(p : q : r) \neq K$ as pole of the isoconjugation which is therefore not isogonal conjugation.
We call $\mathcal{C}_\infty$ the circumconic which is the isoconjugate of $\mathcal{L}^\infty$ and whose equation is :

$$\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 0 \iff pyz + qzx + rxy = 0.$$

$\mathcal{C}_\infty$ is the circum-conic with perspector Ω. Its center is $c_\Omega = [p(q + r - p)] = G/\Omega$.
$\mathcal{C}_\infty$ intersects the circumcircle at A, B, C and a fourth point S_Ω which is the trilinear pole of the line $K\Omega$. S_Ω is analogous to the Steiner point (obtained when $\Omega = G$) and its coordinates are $[1/(b^2r - c^2q)]$.
Now we call δ_Ω the line which is the isoconjugate of the circumcircle and whose equation is :

$$\frac{a^2}{p}x + \frac{b^2}{q}y + \frac{c^2}{r}z = 0 \iff a^2qrx + b^2rpy + c^2pqz = 0.$$

δ_Ω is clearly $\mathbb{P}(\mathbf{tg}\Omega)$. δ_Ω is analogous to the de Longchamps line (obtained when $\Omega = G$).

4.2.1 Pivotal non-isogonal circular cubics

- **Theorem : for any point $\Omega \neq K$, there is one and only one pivotal (non-isogonal) circular $p\mathcal{K}$ with pole Ω.**

  Its pivot is :
  $$P_\Omega = \left[b^2c^2p(q+r-p) - (b^4rp + c^4pq - a^4qr)\right]$$
  which is the reflection of S_Ω in δ_Ω.

  The two circular points at infinity being $\mathcal{J}_1, \mathcal{J}_2$ and their isoconjugates being $\mathcal{J}_1^*, \mathcal{J}_2^*$, it is interesting to remark that the pivot P_Ω is the intersection of the imaginary lines $\mathcal{J}_1\mathcal{J}_1^*$ and $\mathcal{J}_2\mathcal{J}_2^*$. Note that $\mathcal{J}_1^*, \mathcal{J}_2^*$ are the common (imaginary) points of $\mathbb{P}(\mathbf{tg}\Omega)$ and $\mathcal{C}_\infty$.

  Note also that the isoconjugate P_Ω^* of the pivot P_Ω is $\mathbf{ig}P$, the inverse (in the circumcircle of ABC) of the isogonal conjugate of P_Ω. It follows that, if $P_\Omega \neq H$ is given, then the pole Ω is the barycentric product of P and $\mathbf{ig}P$.

  With $P = u : v : w$, this gives
  $$\Omega = [a^2(2u(-S_Au + S_Bv + S_Cw) - (-a^2vw + b^2wu + c^2uv))].$$

  The Droussent cubic is a good example when we consider the isotomic conjugation. See [20]. See also §4.3.5 below.

- **Remarks :**

  1. recall that P_Ω is not defined when $\Omega = K$.
  2. the mapping $\theta\ : \Omega \mapsto P_\Omega$ has four fixed points A, B, C and X_{67}. See §4.3.5 below.
  3. θ maps any point on the orthic axis to H. Hence, all the circular $p\mathcal{K}$ having their pole on the orthic axis have the same pivot H and form a pencil of cubics passing through A, B, C, H, the feet of the altitudes and the circular points at infinity.

- **Constructions**

  The construction met in §1.4.3 is easily adapted in this particular situation and the construction of the real asymptote is now possible with ruler and compass. If T_Ω denotes the reflection of S_Ω about c_Ω and N_Ω^* the isoconjugate of N_Ω (reflection of P_Ω about T_Ω, apoint on the polar conic of P_Ω), then the real asymptote is the parallel at N_Ω^* to the line $P_\Omega T_\Omega$. T_Ω^* is the (real) point at infinity on the cubic and T_Ω obviously lies on the cubic and on $\mathcal{C}_\infty$ as well. It is the coresidual of P_Ω, P_Ω^* and the circular points at infinity. This means that any circle through P_Ω and P_Ω^* meets the cubic at two other points collinear with T_Ω.

  In particular, the circles $AP_\Omega P_\Omega^*$, $BP_\Omega P_\Omega^*$, $CP_\Omega P_\Omega^*$ intersect the lines AT_Ω, BT_Ω, CT_Ω again at A_2, B_2, C_2 respectively, these points lying on the cubic. Hence the parallels to the asymptote at A, B, C meet the lines $P_\Omega A_2$, $P_\Omega B_2$, $P_\Omega C_2$ at α, β, γ respectively, these points lying on the cubic too. A_2 is the isoconjugate of α and can also be seen as the second intersection of circles $P_\Omega^* BC$ and $P_\Omega^* P_\Omega A$.

  It is now possible to construct the point X where the cubic intersects its asymptote. The circles $P_\Omega A\alpha$, $P_\Omega B\beta$, $P_\Omega C\gamma$ meet at X^*. X is the isoconjugate of X^* such that P_Ω, X, X^* are collinear on the curve. We remark that X is the common tangential of A_2, B_2, C_2 and X lies on the circumcircle of $A_2B_2C_2$. This circle passes through the center of the polar conic of the infinite point T^*_Ω, a point on the real asymptote. This conic is actually a rectangular hyperbola that meets the cubic again at four other points which are the in/excenters of $A_2B_2C_2$ and therefore the centers of anallagmaty of the cubic.

This gives the following

- **Theorem : a non-isogonal pivotal circular cubic is an isogonal pivotal circular cubic with respect to the triangle $A_2B_2C_2$.**

The singular focus F of this circular cubic is the common point of the perpendiculars at A_2, B_2, C_2 to the lines XA_2, XB_2, XC_2 (see [5], tome 3, p.90, §21) and F is the antipode of X on the circle $A_2B_2C_2$. F also lies on the perpendicular bisector of $P_\Omega X^*$.

Let us denote by $P_aP_bP_c$ the cevian triangle of P_Ω. The lines P_aA_2, P_bB_2, P_cC_2 concur on the cubic at Q whose isoconjugate Q^* is the sixth intersection of the cubic with the circumcircle. Q is the real intersection of the cubic with δ_Ω and the line QP^*_Ω is parallel to the asymptote.

Note that P_Ω, P^*_Ω, P_Ω/P^*_Ω, Q^* and P_Ω, P^*_Ω, Q, T_Ω are two sets of concyclic points.

Figure 4.4 shows the Droussent cubic **K008** considered as an isogonal circular pivotal cubic.

In this case, $A_2 = -a^2 - b^2 - c^2 : a^2 + b^2 - 2c^2 : a^2 - 2b^2 + c^2$, B_2 and C_2 likewise.

More generally, with $\Omega = p : q : r$, denote

$$U_O = 2\,(S_A\,p + S_B\,q + S_C\,r),$$

$$U_A = 2a^2qr - b^2r(p+q-r) - c^2q(p-q+r),$$
$$T_A = \sqrt{16\Delta^2qr + (c^2q - b^2r)^2}.$$

The corresponding quantities U_B, U_C and T_B, T_C are defined cyclically.

With these notations, we obtain :

$$A_2 = p\,U_O : U_C : U_B \quad ; \quad B_2 = U_C : q\,U_O : U_A \quad ; \quad C_2 = U_B : U_A : r\,U_O,$$

and the incenter of $A_2B_2C_2$ is

$$E_o = p\,U_OT_A + U_CT_B + U_BT_C : q\,U_OT_B + U_AT_C + U_CT_A : r\,U_OT_C + U_BT_A + U_AT_B.$$

The excenters E_a, E_b, E_c are easily obtained by successively replacing in E_o the quantities T_A, T_B, T_C by their opposites.

For example,

$$E_a = -p\,U_OT_A + U_CT_B + U_BT_C : q\,U_OT_B + U_AT_C - U_CT_A : r\,U_OT_C - U_BT_A + U_AT_B.$$

Naturally, with $\Omega = X_6$, the points A_2, B_2, C_2 are the vertices of ABC and the points E_o, E_a, E_b, E_c are its in/excenters.

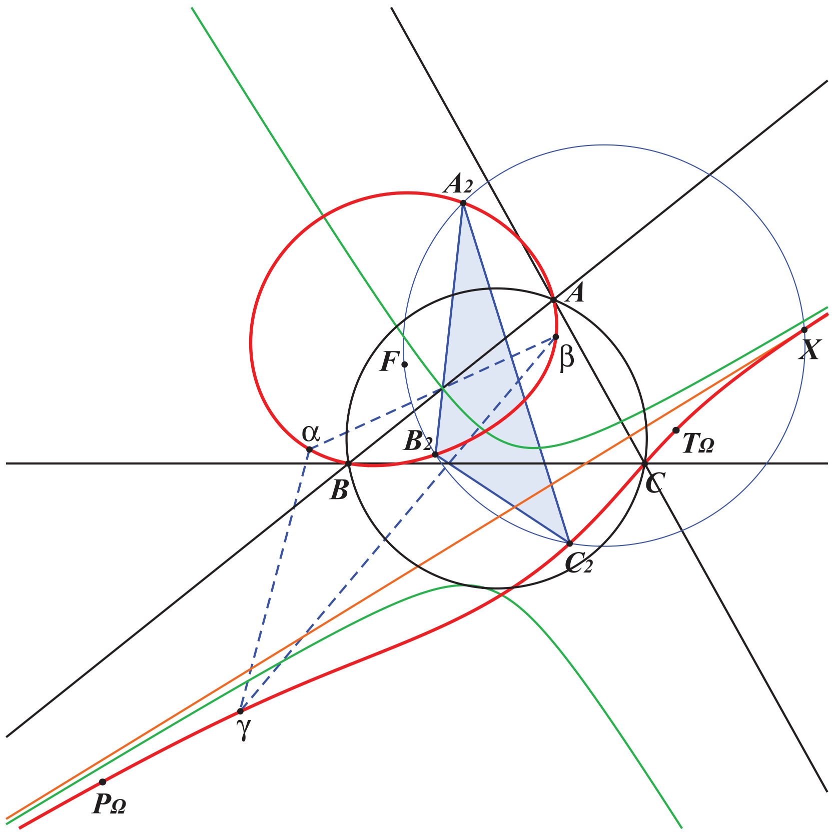

Figure 4.4: The Droussent cubic

- **Theorem : a non-isogonal pivotal circular cubic is a focal if and only if its pole lies on one symmedian of triangle ABC.**

  We already know that the singular focus F of such a cubic lies on a circle having six common points with the curve : A_2, B_2, C_2, X and the circular points at infinity. Hence, F must be one of the points A_2, B_2, C_2 in order to have a focal cubic. Taking for example $F = A_2$, a straightforward but tedious computation shows that the pole Ω must lie on the symmedian AK (Ω distinct of A and K).

  In this case, P_Ω is a point of the altitude AH and the cubic passes through its foot H_a on BC. P_Ω^* lies on the line AO and the cubic is tangent at A to this line. We have $S_\Omega = A$ and δ_Ω is parallel to BC.

4.2.2 Non-pivotal non-isogonal circular cubics

- **Theorem : for any point $\Omega \neq K$, all non-pivotal (non-isogonal) circular cubics with pole Ω form a pencil of cubics.**

  The pencil is generated by the two following degenerate cubics :

  – one into $\mathcal{L}^\infty$ and the circumconic $\mathcal{C}_\infty$ with center c_Ω, the $G-$Ceva conjugate of Ω,

  – the other into the circumcircle and the line δ_Ω with singular focus the circumcenter O.

  The nine base-points defining the pencil are :

– the three vertices of ABC,

– the two circular points at infinity $\mathcal{J}_1, \mathcal{J}_2$,

– their isoconjugates $\mathcal{J}_1^*, \mathcal{J}_2^*$, on δ_Ω and on $\mathcal{C}_\infty$,

– the point S_Ω on the circumcircle, the trilinear pole of the line $K\Omega$, and its isoconjugate $S_\Omega^* = [p(b^2r - c^2q)]$ on $\mathcal{L}^\infty$, the infinite point of δ_Ω.

The root R_Ω of each cubic of the pencil lies on the line through G and $\mathbf{tg}\Omega$. Hence R_Ω^* lies on the circumconic passing through K and Ω. In particular, when $R_\Omega = G$ we have $R_\Omega^* = \Omega$ and when $R_\Omega = \mathbf{tg}\Omega$ we have $R_\Omega^* = K$. The two corresponding cubics are the degenerate cubics above.

The singular focus F_Ω lies on the line passing through O, P_Ω^* (see below) and the isogonal conjugate $\mathbf{g}P_\Omega$ of P_Ω.

The polar conic of S_Ω^* is a rectangular hyperbola for any R_Ω and, when R_Ω traverses the line through G and $\mathbf{tg}\Omega$, these hyperbolas belong to a same pencil generated by the two following decomposed rectangular hyperbolas :

– one is the union of δ_Ω and the perpendicular at O to δ_Ω,

– the other is the union of $\mathcal{L}^\infty$ and the line passing through the midpoint Z_Ω of $P_\Omega S_\Omega$ and the center c_Ω of $\mathcal{C}_\infty$.

Note that Z_Ω lies on the Simson line of S_Ω.

Special cubics of the pencil

For a given pole $\Omega \neq K$, the pencil contains :

– one focal cubic when R_Ω is the orthocorrespondent $P_\Omega^\perp$ of P_Ω. This is the cubic of the pencil passing through P_Ω and P_Ω^*. See below for more details.

– the two decomposed cubics cited above with roots G and $\mathbf{tg}\Omega$ respectively.

– one $n\mathcal{K}_0$.

– one proper $n\mathcal{K}^+$.

Construction of a non-pivotal non-isogonal circular cubic with given pole $\Omega \neq K$

We recall and adapt the construction of this cubic from that of §1.5.4 since we know two isoconjugate points on the cubic namely S_Ω and S_Ω^*.

With given root R_Ω on the line G-$\mathbf{tg}\Omega$, we construct the trilinear polar l_R of R_Ω meeting the sidelines of ABC at U, V, W. Note that, when R_Ω varies, l_R envelopes the inscribed parabola with perspector the trilinear pole of the line G-$\mathbf{tg}\Omega$. This parabola has focus S_Ω and directrix the radical axis of the circles with diameters AU, BV, CW. This directrix passes through P_Ω.

The cevian lines of S_Ω meet l_R at U_1, V_1, W_1. For any M on l_R, we construct the homologue M' of M in the involution that swaps U and U_1, V and V_1, W and W_1. The line $S_\Omega M'$ meets the circumconic passing through S_Ω^* and M^* at two points on the cubic.

In particular, the cubic contains $A_1 = S_\Omega A \cap S_\Omega^* U$ and its isoconjugate $A_1^* = S_\Omega U \cap S_\Omega^* A$, four analogous points being defined similarly. Note that six lines are parallel to the real asymptote namely UA_1, AA_1^*, etc, and the conic passing through the midpoints of any two of these six pairs of points is the rectangular hyperbola that is the polar conic of the infinite point S_Ω^* of the cubic. Thus the real asymptote of the cubic passes through the center of this rectangular hyperbola.

When the involution is hyperbolic, it has two real fixed points F_1, F_2 on the line l_R. In this case, the lines $S_\Omega F_1$, $S_\Omega F_2$ meet the polar conic of S_Ω^* at four points and the cubic is an isogonal pivotal cubic with respect to the diagonal triangle of the quadrilateral formed by these four points. The pivot is S_Ω^* and the four points are the in/excenters of the diagonal triangle. One vertex is obviously S_Ω and the two other lie on the polar line of S_Ω in the polar conic of S_Ω^*. This polar line contains P_Ω and the third point of the cubic on this line is its intersection X^* with the line $S_\Omega S_\Omega^*$, the parallel at S_Ω to the asymptote.

The circumcircle of the diagonal triangle contains the center of the polar conic of S_Ω^* and meets the real asymptote again at X on the cubic. X is the isopivot of the cubic i.e. the isogonal conjugate of S_Ω^* with respect to the diagonal triangle.

The antipode of X on the circle is the singular focus F of the cubic and the parallel to the real asymptote at the center of the circle (the midpoint of XF) is the orthic line of the cubic.

Characterization of a non-pivotal non-isogonal circular cubic with given pole $\Omega \neq K$

Following §1.5.3, any non-pivotal non-isogonal circular cubic with given pole $\Omega \neq K$ is the locus of M such that M and M^* are conjugated with respect to a circle with center P_Ω. This circle is orthogonal to all the circles with diameter any two isoconjugate points on the cubic such as AU, BV, CW, $A_1A_1^*$, etc.

- **Theorem : for any point $\Omega \neq K$, there is one and only one non-pivotal (non-isogonal) circular focal cubic with pole Ω.**

Recall that its root is the orthocorrespondent $P_\Omega^\perp$ of P_Ω.

Its singular focus is P_Ω^* on the curve that also passes through P_Ω.

The point $X^* = S_\Omega S_\Omega^* \cap P_\Omega P_\Omega^*$ is the isoconjugate of X intersection of the cubic with its real asymptote. We remark that X is the antipode of P_Ω^* on the circle through P_Ω, P_Ω^* and S_Ω. This asymptote is clearly the homothetic of the line δ_Ω, center P_Ω^*, ratio 2.

The polar conic of X passes through S_Ω, S_Ω^*, P_Ω, P_Ω^* and X : the tangents to the cubic at S_Ω, S_Ω^*, P_Ω, P_Ω^* all pass through X. It follows that this focal cubic is an isogonal circular pivotal cubic with respect to the triangle $P_\Omega P_\Omega^* S_\Omega$.

This focal cubic can be seen as the locus of contacts M and N of the tangents drawn from P_Ω^* to the circles of the pencil generated by the two circle-points P_Ω and S_Ω. The Poncelet points of this pencil of circles are the points $\mathcal{J}_1^*, \mathcal{J}_2^*$ on δ_Ω. If M^* and N^* are the isoconjugates of M and N, we have the four following collinearities on the cubic :

$$X, M, N - X, M^*, N^* - X^*, M^*, N - X^*, M, N^*.$$

The polar conic of P_Ω^* is the circle of this pencil that passes through P_Ω^*.

An example of such cubic is given in §4.3.6 with isotomic conjugation.

Construction

The construction of the focal cubic is fairly easy. A variable line passing through P^*_Ω and the center of a variable circle passing through P_Ω and S_Ω meets the circle at two points on the focal cubic.

Characterization of a non-pivotal non-isogonal focal cubic with given pole $\Omega \neq K$

Any non-pivotal non-isogonal circular cubic with given pole $\Omega \neq K$ is the locus of M such that M and M^* are conjugated with respect to the circle with center P_Ω and radius 0. In other words, this cubic is the locus of M such that the circle with diameter MM^* contains P_Ω.

4.3 Some examples

4.3.1 K060 $= \mathcal{K}_n$, K073 $= \mathcal{K}_i$: two circular pivotal cubics

It is known that the Neuberg cubic is the locus of point M such that the triangles ABC and $M_aM_bM_c$ are in perspective, where M_a, M_b, M_c are the reflections of M about the sidelines of ABC [5]. This means that the lines AM_a, BM_b, CM_c are concurrent at N and then the locus of N is the cubic $\mathcal{K}_n$ or **K060** . See figure 4.5.
Its equation is :

$$\sum_{\text{cyclic}} S_A\, x\, \left[(4S_C^2 - a^2b^2)z^2 - (4S_B^2 - c^2a^2)y^2\right] = 0$$

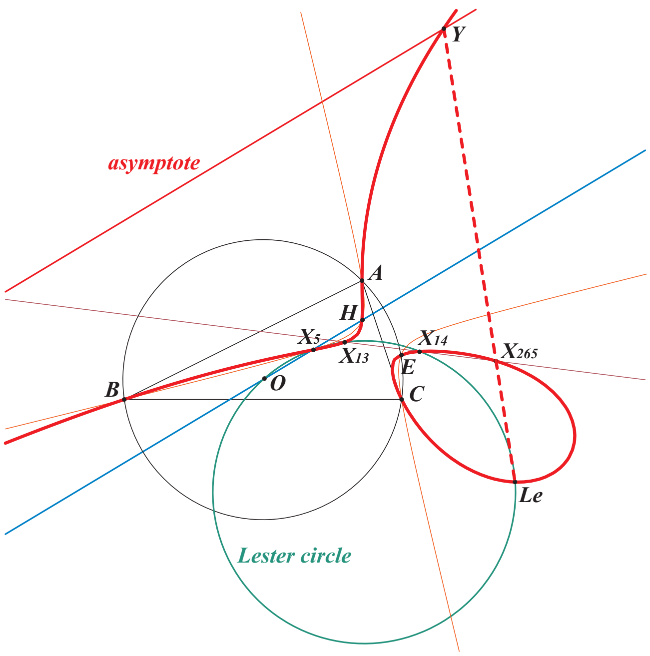

Figure 4.5: **K060** or the $\mathcal{K}_n$ cubic

[5] $M_aM_bM_c$ is called 2-pedal triangle in [51] and $\mathcal{K}_n$ is the 2-cevian cubic associated to the 2-pedal cubic (Neuberg) and the -2-pedal cubic (Napoleon).

$\mathcal{K}_n$ is the circular $p\mathcal{K}$ with pivot $X_{265} = \mathbf{gi}H$ and pole a point named $P_o = X_{1989} = \mathbf{g}X_{323}$ [6]. This point will have a great importance in the following chapters.
$\mathcal{K}_n$ passes through many centers such that H (the isoconjugate of the pivot), X_5, X_{13}, X_{14}, X_{30}, X_{79}, X_{80}, X_{265}, X_{621}, X_{622} [7] and is tangent at A, B, C to the altitudes of triangle ABC.
Its singular focus (not on the curve) is the reflection of X_{110} about X_{125} or the reflection of X_{399} about X_5. [8]
The fourth real intersection E with the circumcircle is the second intersection of the line X_5X_{110} with the circumcircle and also the sixth intersection with the rectangular hyperbola through A, B, C, H, X_5. [9]
Its real asymptote is parallel to the Euler line at X_{399} or X_{323}. $\mathcal{K}_n$ intersects its asymptote on the line through X_{265} and $\mathbf{gig}X_{399}$, this point lying on the Lester circle and on $\mathcal{K}_n$.
$\mathcal{K}_n$ is also the isogonal image of the inversive image of the Neuberg cubic in the circumcircle. Hence any point on the Neuberg cubic gives another point on $\mathcal{K}_n$.
$\mathcal{K}_n$ can also be obtained as the locus of point Q center of perspective of ABC and $A_MB_MC_M$ where M is on the Neuberg cubic and A_M is the isogonal conjugate of A with respect to triangle MBC (B_M and C_M defined similarly)

Remark :

The inversive image of the Neuberg cubic in the circumcircle is another circular $p\mathcal{K}$ we shall denote by $\mathcal{K}_i$ or **K073**.
Its equation is :

$$\sum_{\text{cyclic}} a^2 S_A\, x\ \left[c^4(4S_C^2 - a^2b^2)y^2 - b^4(4S_B^2 - c^2a^2)z^2\right] = 0$$

Its pivot is O and its pole is X_{50}.
It passes through $O, X_{15}, X_{16}, X_{35}, X_{36}, X_{54}, X_{186}$, the midpoint of OX_{110} (which is $\mathbf{i}X_{399}$), the point at infinity of the Euler line of the orthic triangle (isoconjugate of X_{54}) and obviously the inverses of all the points lying on the Neuberg cubic.

4.3.2 Isogonal circum-strophoids. The Pelletier strophoid K040

- Following §4.1.2, we find a pencil of isogonal strophoids with double point at I (incenter) if and only if the root P is on the trilinear polar of the Gergonne point (X_7 in [38, 39]) with equation :

$$(b + c - a)x + (c + a - b)y + (a + b - c)z = 0$$

- One case is particularly interesting : when P is X_{514} (point at infinity of the trilinear polar above), we obtain the isogonal strophoid with equation :

$$\sum_{\text{cyclic}} (b - c)x(c^2y^2 + b^2z^2) + 2(b - c)(c - a)(a - b)xyz = 0$$

It is the locus of foci of inscribed conics whose center is on the line IG.

---

[6] X_{323} is the reflection of X_{23} about X_{110}, where $X_{23} = \mathbf{i}G$ and X_{110} is the focus of the Kiepert parabola. P_o is also the barycentric product of the Fermat points.
[7] X_{621}, X_{622} are the anticomplements of the isodynamic points X_{15}, X_{16} resp.
[8] This point is $E_{2027} = [2(S_B\, b^4 + S_C\, c^4 - S_A\, a^4) - a^2b^2c^2]$.
[9] This point is $X_{1141} = 1/\left[(4S_A^2 - b^2c^2)[a^2(b^2 + c^2) - (b^2 - c^2)^2]\right]$

It passes through X_1, X_{36}, X_{80}, X_{106} (singular focus), X_{519} and the foci of the Steiner inscribed ellipse.The nodal tangents are parallel to the asymptotes of the Feuerbach rectangular hyperbola.

- Another case is worth noticing : when P is the perspector of the Feuerbach hyperbola i.e. the intersection of all the trilinear polars of the points of the Feuerbach hyperbola. This point is X_{650} and is sometimes called Pelletier point with coordinates : $[a(b-c)(b+c-a)]$. The trilinear polars of I, H, the Gergonne and the Nagel points and many more all pass through it.

  We shall call Pelletier strophoid the circum-strophoid with root at this point. See figure 4.6.

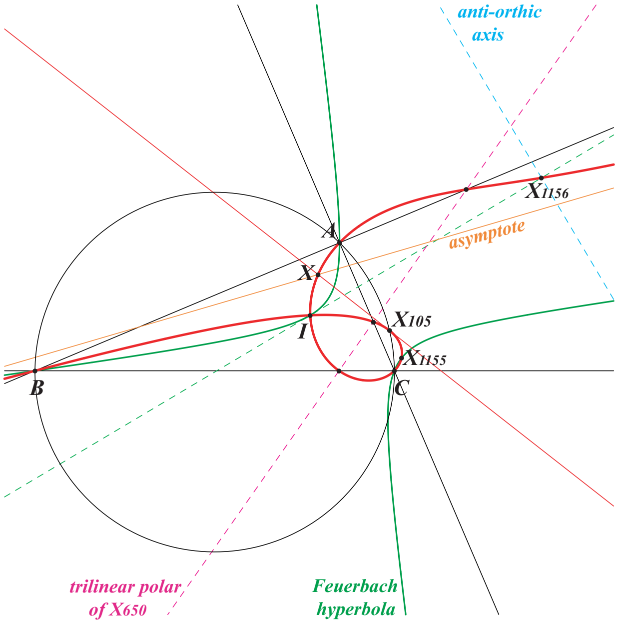

Figure 4.6: **K040** the Pelletier strophoid

Its equation is :

$$\sum_{\text{cyclic}} a(b-c)(b+c-a)\; x\; (c^2y^2+b^2z^2)=0$$

Its singular focus is $X_{105} = \mathbf{g}X_{518}$ (X_{518} is the point at infinity of the line IK).

This cubic is the locus of foci of inscribed conics whose center is on the line IK.

X_{243} and X_{296} are two isogonal conjugates lying on it.

Another point on the cubic is the intersection E of OI with the anti-orthic axis.

It is X_{1155} in [39] with coordinates :

$$\left[a\left(b^2+c^2-2a^2+ab+ac-2bc\right)\right] = \left[a\left((b-c)^2+a\left(b+c-2a\right)\right)\right]$$

$\mathbf{g}E = X_{1156}$ also lies on the cubic and is the last common point with the Feuerbach hyperbola.

- The construction of the isogonal circum-strophoid $\mathcal{S}_\ell$ with double point at I and a real asymptote parallel to the given line ℓ is fairly easy :

First, construct the isogonal conjugate F of the point at infinity of ℓ [10] (F is the singular focus) and the circle (γ) centered at I passing through F. Then, draw the rectangular hyperbola $\mathcal{H}_\ell$ through I, F and the three inversive images of A, B, C with respect to (γ). If M is on $\mathcal{H}_\ell$, the locus of its inversive image in (γ) is $\mathcal{S}_\ell$. (See [41] for more informations)

Another very simple technique (due to McLaurin) consists in drawing the line Δ parallel to ℓ at F as defined above and take the symmetric F' of I about F. Then, for any point m on Δ, let M be the intersection of the perpendicular at m to $F'm$ and the parallel at I to $F'm$. The locus of M is $\mathcal{S}_\ell$.

From this it is clear that $\mathcal{S}_\ell$ is the pedal curve with respect to I of the parabola with focus at F' and directrix the parallel at I to ℓ. Hence for any point M on $\mathcal{S}_\ell$, the line through M and $\mathbf{g}M$ is tangent to the parabola. $\mathcal{S}_\ell$ is an isogonal conico-pivotal (see [24] and §8) cubic with pivotal conic the parabola described above.

The real asymptote is the reflection of Δ about I. The projection of I on Δ is on $\mathcal{S}_\ell$ and its isogonal conjugate is the intersection of $\mathcal{S}_\ell$ with its asymptote. The circle with diameter IF' intersects Δ at two points which are on the two tangents at I. The perpendicular at F to IF intersects the bisectors of the tangents at I at two points where the tangents are parallel to the asymptote. They are the two centers of anallagmacy of $\mathcal{S}_\ell$.

Finally, the circle centered at I through F intersects $\mathcal{S}_\ell$ at F and three other points which are the vertices of an equilateral triangle.

Remark :

If ℓ is parallel to a bisector of ABC, say AI, $\mathcal{S}_\ell$ degenerates into the union of this bisector and the circle through B, C and the excenter which is on AI.

4.3.3 Three circular isogonal Brocard cubics

- **K018** $= \mathcal{B}_2$

  (We start from §4.1.2 again) If P is the point at infinity of the orthic axis [11] (X_{523} in [39]), we find a nice Van Rees focal denoted by $\mathcal{B}_2$ with focus at the Parry point (X_{111} in [38, 39]) and its asymptote parallel to the line GK. See figure 4.7.

  The cubic passes through G, K, the Fermat points X_{13}, X_{14}, the isodynamic points X_{15}, X_{16}, the equi-Brocard center X_{368}[12], its isogonal conjugate, X_{524} (point at infinity of GK), the vertices of the second Brocard triangle and their isogonal conjugates.

  Its equation is :

$$\sum_{\text{cyclic}} (b^2 - c^2)\, x\, (c^2y^2 + b^2z^2) = 0$$

  $\mathcal{B}_2$ is the locus of :

  1. contacts of tangents drawn from X_{111} to the circles passing through G and K.

[10] Draw a parallel at A to ℓ and reflect this parallel about the bisector AI. F is its second intersection with the circumcircle.

[11] This point is the perspector of the Kiepert hyperbola.

[12] See [38], p.267

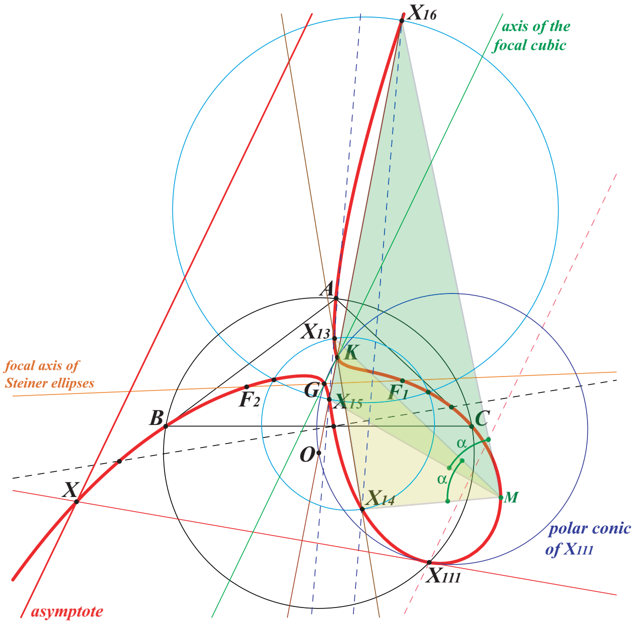

Figure 4.7: **K018** = $\mathcal{B}_2$ the second Brocard cubic

2. foci of inscribed conics whose center is on the line GK. From this, it is obvious that $\mathcal{B}_2$ passes through the foci of the inscribed Steiner ellipse.
3. point M such that the pole of the line $M\mathbf{g}M$ in the circum-conic through M and $\mathbf{g}M$ lies on the Brocard axis OK. (See proposition 2 in §1.5.2)
4. point M such that K, M and the orthocorrespondent of M are collinear. (See footnote in §7.2.2)
5. point M such that the three circles MBC, MCA, MAB meet the sidelines of triangle ABC again at six points lying on a same conic.

- **K019** = $\mathcal{B}_3$

If P is the point $[a^2(b^2 - c^2)S_A]$ [13], we get another Van Rees focal denoted by $\mathcal{B}_3$ with focus at the Tarry point (X_{98} in [38, 39]) with its asymptote parallel to the line OK. $\mathcal{B}_3$ passes through the two Brocard points Ω_1, Ω_2. Its equation is :

$$\sum_{\text{cyclic}} a^2(b^2 - c^2)S_A\, x\, (c^2y^2 + b^2z^2) = 0$$

$\mathcal{B}_3$ is the locus of :

1. intersections of a circle through $\Omega_1\Omega_2$, whose center is Ω, with the line $\Omega\, X_{98}$.
2. contacts of the tangents drawn from X_{98} to the circles centered on $\Omega_1\Omega_2$ and orthogonal to the circle with diameter $\Omega_1\Omega_2$.
3. foci of inscribed conics whose center is on the line OK.
4. point M such that the pole of the line $M\mathbf{g}M$ in the circum-conic through M and $\mathbf{g}M$ lies on the Euler line.

[13] This point is X_{647} in [38, 39]. It lies on the orthic axis. It is the perspector of the Jerabek hyperbola. It is also the isogonal conjugate of the trilinear pole of the Euler line

- **K021** $= \mathcal{B}_5$: **Equal-area cevian triangles cubic**

  In a posting to the Math Forum dated Nov. 2,1999 (see also [40]), Clark Kimberling raised and solved the following problem : the locus of point M such that the cevian triangles of M and $\mathbf{g}M$ have the same area is the cubic denoted by $\mathcal{B}_5$ with equation :

$$\sum_{\text{cyclic}} a^2(b^2-c^2)x(c^2y^2-b^2z^2)=0$$

  This is clearly an isogonal circular $p\mathcal{K}$ with pivot X_{512} (point at infinity of the Lemoine axis), with singular focus X_{98} (Tarry point) not on the curve. The cubic passes through X_{99} (Steiner point $= \mathbf{g}X_{512}$) and the two Brocard points.

- **Remark :**

  There are two other "Brocard" isogonal (non circular) cubics :

  – **K017** $= \mathcal{B}_1$ is a $n\mathcal{K}$ through G, K, X_{99} = Steiner point, $X_{512} = \mathbf{g}X_{99}$, the vertices of the first Brocard triangle and their isogonal conjugates.
  Its equation is :

$$\sum_{\text{cyclic}} (a^4-b^2c^2)\ x\ (c^2y^2+b^2z^2)=0$$

  – **K020** $= \mathcal{B}_4$ is a $p\mathcal{K}$ with pivot X_{384}[14] through many centers such that I, O, H, X_{32}, X_{39}, X_{76}, X_{83}, X_{194}, the vertices of the first Brocard triangle and their isogonal conjugates.
  Its equation is :

$$\sum_{\text{cyclic}} (a^4+b^2c^2)\ x\ (c^2y^2-b^2z^2)=0$$

4.3.4 K072 another remarkable Van Rees focal

We know that there is a unique isogonal $p\mathcal{K}$ through the points G, O, H, K and it is the Thomson cubic with pivot G.
There is also a unique isogonal $n\mathcal{K}$ through the same points and it is a Van Rees focal with singular focus $F = X_{842}$.[15] See figure 4.8.

This cubic is the locus of point M such that the segments GO and KH are seen under equal or supplementary angles. It is also the locus of point P whose pedal circle is centered on the parallel at X_5 to the line GX_{98}.

[14] X_{384} is center of perspective of the first Brocard triangle and the triangle formed by the isogonal conjugates. This point is on the Euler line.

[15] F is also :

- the reflection of X_{98} (Tarry point) about the Euler line.
- the reflection of $X_{74} = \mathbf{g}X_{30}$ about the Brocard line.
- the second intersection (apart from X_{112}) of the circumcircle and the circle OHK.
- the second intersection (apart from X_{110}) of the circumcircle and the circle OGX_{110}, isogonal conjugate of X_{542}, point at infinity of the Fermat line i.e. the line through the Fermat points X_{13}, X_{14}.

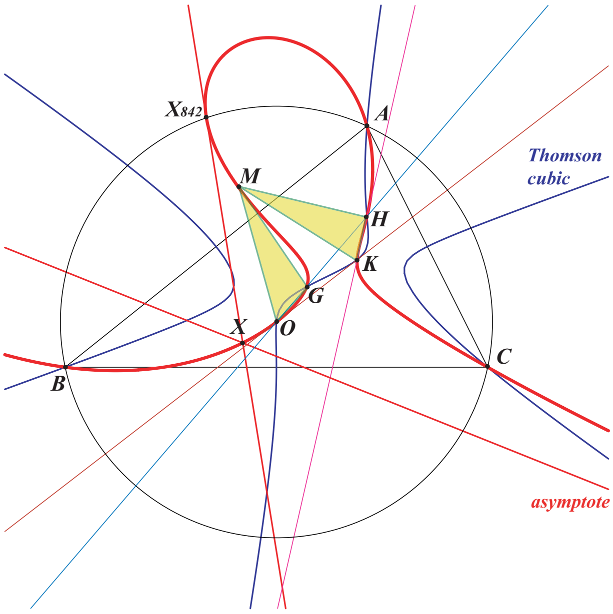

Figure 4.8: **K072** a Van Rees focal through G, O, H, K

Its equation is :

$$\sum_{\text{cyclic}} (b^2 - c^2)(b^4 + c^4 - a^4 - b^2c^2)\, x\, (c^2y^2 + b^2z^2)$$
$$-(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)(a^2 + b^2 + c^2)\, xyz = 0$$

4.3.5 K103 parallel trilinear polars circular $p\mathcal{K}$

Following the remark in §1.4.2, it is easy to see that there exists only one circular $p\mathcal{K}$ locus of point M such that the trilinear polars of M and its Ω-isoconjugate M^* are parallel. The pivot and the pole are X_{67}, isotomic conjugate of the Droussent pivot X_{316}. Its equation is :

$$\sum_{\text{cyclic}} (b^4 + c^4 - a^4 - b^2c^2)\, x^2\, (y - z) = 0$$

This cubic passes through G, X_{67}, X_{141}, X_{524} and is tangent at A, B, C to the medians. Its real asymptote is parallel to GK.
The "last" intersection with the circumcircle is the point :

$$\left[\frac{1}{(b^2 + c^2)(b^4 + c^4 - a^4 - b^2c^2)}\right]$$

The X_{67}-isoconjugate X^*_{524} of X_{524} and the tangential $\widetilde{G}$ of G are other simple points on the curve. Their coordinates are :

$$X^*_{524} = \left[\frac{1}{(b^2 + c^2 - 2a^2)(b^4 + c^4 - a^4 - b^2c^2)}\right]$$

and

$$\widetilde{G} = \left[b^4 + c^4 - 3a^4 + a^2b^2 + a^2c^2 - b^2c^2\right]$$

4.3.6 K091 the isotomic focal $n\mathcal{K}$

We apply the results found in §4.2.2 when $\Omega = G$.
We find that all isotomic circular $n\mathcal{K}$ form a pencil and pass through X_{99} (Steiner point) and X_{523} : the real asymptote is perpendicular to the Euler line. The root lies on the line $G\mathbf{t}K$ and the singular focus on the line OX_{67}.
The most remarkable is the one with focus X_{67} since it is the only focal of the pencil and is closely related to the Droussent cubic, X_{67} being the isotomic conjugate of the Droussent pivot X_{316}, those two points lying on the curve. See figure 4.9. The perpendicular at X_{99} to the Euler line intersects the line $X_{67}X_{316}$ at the isotomic conjugate of the point where the cubic meets its asymptote. Those two points are unknown in [38, 39].
Its root is :

$$(a^2+b^2+c^2)X_{76} + 2(4\,S_AS_BS_C + a^2b^2c^2)X_2 = [p_a : p_b : p_c]$$

where $X_{76} = \mathbf{t}K = [b^2c^2]$ and $X_2 = [1]$. This cubic has equation :

$$\sum_{\text{cyclic}} p_a x(y^2+z^2) - 2\left(\sum_{\text{cyclic}} a^4(2b^2+2c^2-a^2)\right) xyz = 0$$

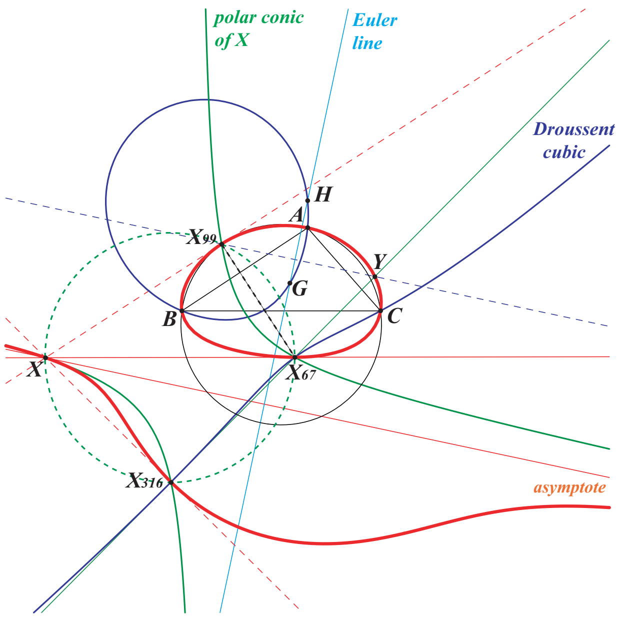

Figure 4.9: K091 the isotomic focal $n\mathcal{K}$ and the Droussent cubic

4.4 Self-inverse circular isocubics in the circumcircle or $p\mathcal{K}^\circ(P)$

4.4.1 Generalities

We know (see [5], tome 3, p.88 for example) that an inversion transforms a circular cubic into a bicircular quartic when the pole Π of the inversion is not on the curve and into another circular cubic passing through Π when the pole is on the curve.

In this paragraph, we seek isocubics (denoted by $\mathcal{K}^{\circ}$) invariant under inversion with respect to the circumcircle which we call **inversible cubics**. Following [5] again, we find that such a cubic must pass through O and the polar conic of its intersection with its real asymptote must pass through O too. Now, since A, B, C are invariant under inversion, the tangents at these points must pass through O and any requested cubic is member of the pencil of cubics through A, B, C, O, the two circular points at infinity with tangents at A, B, C through O. This shows that such a cubic must be a $p\mathcal{K}$ with pivot P on the circumcircle and will now be denoted by $p\mathcal{K}^{\circ}(P)$.
With $P = (p : q : r)$ on the circumcircle, this cubic has equation :

$$\sum_{\text{cyclic}} qr\, x^2\, (c^2 S_C y - b^2 S_B z) = 0$$

This cubic is invariant under the isoconjugation which swaps P and O, whose pole Ω is on the circumconic with perspector X_{184}. [16] This circumconic intersects the circumcircle at A, B, C and X_{112}. For any point P, the line $P\Omega$ passes through X_{112}.
Let us now denote by P_a, P_b, P_c the vertices of the cevian triangle of P. Those three points are obviously on the cubic which is now entirely determined with at least ten of its points : A, B, C, O, P, P_a, P_b, P_c and the two circular points at infinity.
Moreover, the curve is invariant under three other inversions with poles P_a, P_b, P_c swapping A and P, B and P, C and P respectively.
Taking two isoconjugates M and M^* on $p\mathcal{K}^{\circ}(P)$, the points $\mathbf{i}M$ and $\mathbf{i}M^*$ lie on $p\mathcal{K}^{\circ}(P)$ and the line $\mathbf{i}M\mathbf{i}M^*$ passes through a fixed point E of $p\mathcal{K}^{\circ}(P)$. This point E is :

- $\mathbf{gc}P$,
- the point where the parallel at P to the real asymptote $\mathcal{A}$ intersects $p\mathcal{K}^{\circ}(P)$,
- the point at which the lines $A\,\mathbf{i}P_a$, $B\,\mathbf{i}P_b$, $C\,\mathbf{i}P_c$ concur on $p\mathcal{K}^{\circ}(P)$.

From this, we see that the tangents at O, P_a, P_b, P_c are parallel to $\mathcal{A}$ (and to PE) and that the tangents at $\mathbf{i}P_a$, $\mathbf{i}P_b$, $\mathbf{i}P_c$ concur at X which is the point where $p\mathcal{K}^{\circ}(P)$ and $\mathcal{A}$ meet. [17] We notice that X is the isoconjugate of the inverse of E.
The singular focus F of $p\mathcal{K}^{\circ}(P)$ is the point where the normals to $p\mathcal{K}^{\circ}(P)$ at $\mathbf{i}P_a$, $\mathbf{i}P_b$, $\mathbf{i}P_c$ concur. When P sweeps the circumcircle out, the locus of F is the circumcircle of the tangential triangle centered at X_{26} on the Euler line. The circle with diameter XF passes through $\mathbf{i}P_a$, $\mathbf{i}P_b$, $\mathbf{i}P_c$: it is the 9-point circle of triangle $P_aP_bP_c$.
According to known properties of circular cubics, the envelope of the perpendicular bisector of $M\mathbf{i}M$ is a parabola with focus at F and directrix the parallel to $\mathcal{A}$ which is the image of $\mathcal{A}$ under the $\overrightarrow{OF}$-translation. Thus, the cubic is the envelope of bitangent circles centered on the parabola and orthogonal to the circumcircle, the contacts of the circle being two inverse points M and $\mathbf{i}M$.
At last, let us remark that $p\mathcal{K}^{\circ}(P)$ is a focal cubic if and only if P is a reflection of H in one sideline of ABC. The singular focus is the inverse of the foot of the corresponding altitude.

4.4.2 Construction of a $p\mathcal{K}^{\circ}(P)$

Let us first summarize the study above : for any point P on the circumcircle, the triangle $P_aP_bP_c$ formed by its traces has orthocenter O and the feet of the altitudes are

[16] $X_{184} = \mathbf{gt}O$.
[17] Remember that the tangents at M and $\mathbf{i}M$ are symmetric about the perpendicular bisector of $M\mathbf{i}M$.

their inverses $\mathbf{i}P_a$, $\mathbf{i}P_b$, $\mathbf{i}P_c$.
$E = \mathbf{gc}P$ is the point on the curve such that its isoconjugate E^* lies on $\mathcal{L}^\infty$: in other words, the real asymptote is parallel to the line PE.
Let X be the isoconjugate of $\mathbf{i}E$ in the isoconjugation that swaps P and O : X is the point where the cubic meets its real asymptote and the antipode F of X on the circle $\mathbf{i}P_a\mathbf{i}P_b\mathbf{i}P_c$ is the singular focus.
For any point ω on the perpendicular at O to PE, the circle centered at ω passing through O intersects the perpendicular at X to ωF at two points on the cubic.

4.4.3 Several examples of $p\mathcal{K}^\circ(P)$

- **K114** $= p\mathcal{K}^\circ(X_{74})$ is particularly interesting since it belongs to the pencil of circular cubics generated by the Neuberg cubic and the cubic $\mathcal{K}_i$ we met in §4.3.1.

  Their nine common points are A, B, C, O, X_{15}, X_{16}, X_{74} and the circular points at infinity. The point E is here $\mathbf{g}X_{113}$. See figure 4.10.

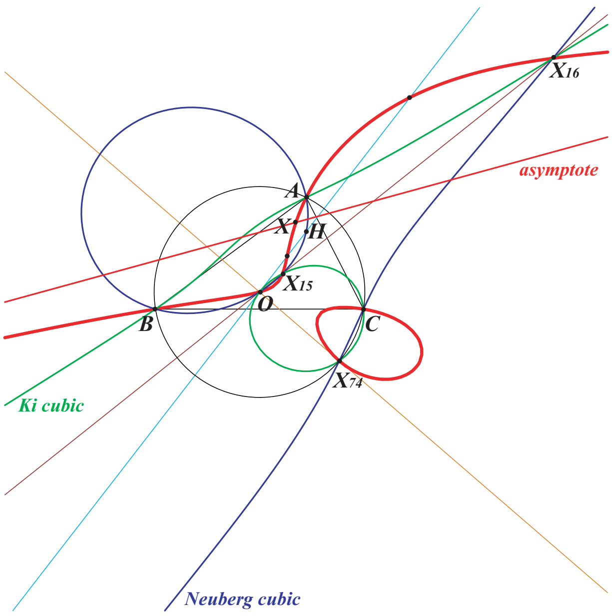

Figure 4.10: **K114** $= p\mathcal{K}^\circ(X_{74})$, $\mathcal{K}_i$ and the Neuberg cubic

- **K112** $= p\mathcal{K}^\circ(X_{1141})$ – where $X_{1141} = E_{368}$ denotes the second intersection of the line through X_5 and X_{110} and the circumcircle – is probably the most remarkable cubic of this type : its nine common points with the Neuberg cubic are A, B, C, O, X_{13}, X_{14}, $E_{389} = X_{1157}$ [18] and the circular points at infinity. It passes also through $X_{54}, X_{96}, X_{265}, X_{539}$ (at infinity), $\mathbf{g}X_{128}$ and the inverses of $X_{13}, X_{14}, X_{96}, X_{265}$ not mentioned in [38, 39]. See figure 4.11.

- **K113** $= p\mathcal{K}^\circ(E_{591})$ – where $E_{591} = \mathbf{t}X_{858} = X_{2373}$ [19] – has nine points in common with the Droussent cubic : A, B, C, O, G, X_{69}, X_{524}, E_{591} and the circular points at infinity. $\mathbf{g}X_{858}$ (on the Jerabek hyperbola) and $\mathbf{i}X_{69}$ lie on the cubic as well. See figure 4.12.

[18] $E_{389} = \mathbf{ig}X_5$.
[19] X_{858} is the intersection of the Euler and de Longchamps lines

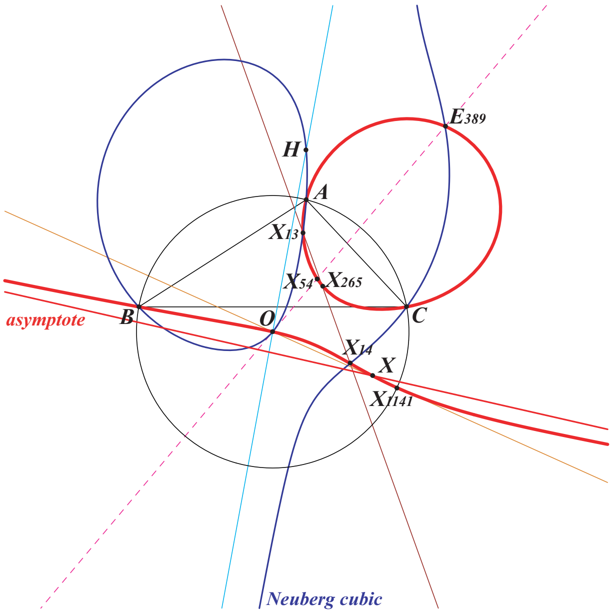

Figure 4.11: **K112** $= p\mathcal{K}^{\circ}(E_{368})$ and the Neuberg cubic

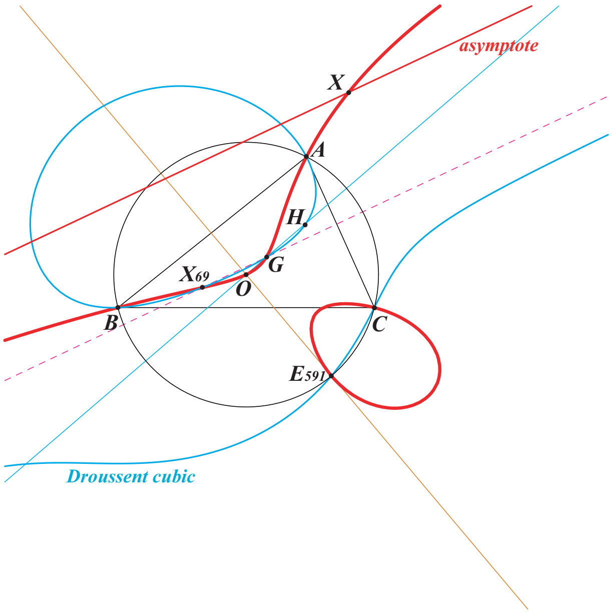

Figure 4.12: **K113** $= p\mathcal{K}^{\circ}(E_{591})$ and the Droussent cubic

Chapter 5

$\mathcal{K}_{60}$ cubics : general theorems

5.1 Theorem 1 for $\mathcal{K}_{60}$ cubics

A cubic $\mathcal{K}$ is a $\mathcal{K}_{60}$ if and only if the polar conic of each point on $\mathcal{L}^\infty$ is a rectangular hyperbola.

Denote that this result is true whatever the cubic is circumscribed or not.

An immediate consequence :

Since all polar conics of the points on $\mathcal{L}^\infty$ belong to the same pencil, we only need to check that two of them are rectangular hyperbolas and it is convenient to choose the points at infinity of ABC sidelines. In this case, the equations of the three polar conics are : $F'_x = F'_y$, $F'_y = F'_z$, $F'_z = F'_x$ which shows their great simplicity for a practical usage.

Proof of Theorem 1 :

- Taking an orthonormal cartesian coordinates system, the axis of y being directed by one of the asymptotic directions of the cubic, let us denote by $u = \infty, v, w$ the slopes (v, w not necessarily real) of those asymptotic directions. Since all polar conics of the points of $\mathcal{L}^\infty$ are in a pencil, those conics are rectangular hyperbolas if and only if the polar conics of the points at infinity of the cubic are themselves rectangular hyperbolas.

  Let us consider now two complex numbers v, w and let $U = \frac{1}{2}(v + w), V = 2w - v, W = 2v - w$. If $u = \infty$, we have :

  $$(v, w, u, U) = (w, u, v, V) = (u, v, w, W) = -1$$

  which means that $(u, U), (v, V), (w, W)$ are the slopes of the asymptotic directions of the polar conics of the points at infinity of the cubic. Those three conics are rectangular hyperbolas if and only if :

  $$(i, -i, u, U) = (i, -i, v, V) = (i, -i, w, W) = -1$$

  $$\iff v + w = 2vw - v^2 + 1 = 2vw - w^2 + 1 = 0 \iff v = -w = \pm\frac{1}{\sqrt{3}}$$

  which is equivalent to the fact that the cubic is $\mathcal{K}_{60}$.

- If three distinct points of a line ℓ have their polar lines concurring at a point M_o, the polar conic of M_o must pass through those three points and therefore is degenerate into two lines, one of them being ℓ. Hence, the polar line of any point of ℓ passes through M_o.

  Now, let $\mathcal{K}$ be a circumcubic having three distinct real asymptotes. Taking $\ell = \mathcal{L}^\infty$, it is clear that its asymptotes concur at M_o if and only if the polar lines of any three distinct points on it (and in particular the points at infinity of AB, BC, CA) concur at M_o.

Remark :

The conic with equation $A_x x^2 + A_y y^2 + A_z z^2 + B_x yz + B_y zx + B_z xy = 0$ is a rectangular hyperbola if and only if $A_x a^2 + A_y b^2 + A_z c^2 = B_x S_A + B_y S_B + B_z S_C$.

5.2 Theorem 2 for $\mathcal{K}_{60}^+$ cubics

A cubic is a $\mathcal{K}_{60}^+$ if and only if the polar conic of each point of the plane is a rectangular hyperbola.

Proof of Theorem 2 :

- If the cubic is a $\mathcal{K}_{60}^+$, its asymptotes concur at M_o whose polar conic degenerates into two lines, one of them being $\mathcal{L}^\infty$. The polar conic of any point M belongs to the pencil of conics generated by the polar conics of M_o and of the point at infinity of the line MM_o. Hence, the polar conics of M and of the point at infinity of the line MM_o have the same points at infinity which shows the former is a rectangular hyperbola since the latter is already one.

- If the polar conics of all points in the plane are rectangular hyperbolas, the cubic is $\mathcal{K}_{60}$. M_o being the common point of two asymptotes, the polar conic C_{M_o} of M_o goes through the corresponding points at infinity. Since the asymptotes make 60° angle, C_{M_o} cannot be a proper rectangular hyperbola : it degenerates and contains $\mathcal{L}^\infty$, thus the third asymptote must pass through M_o.

Immediate consequences :

- The polar conics of all points of the plane form a net of conics. From this and since the knowledge of three non collinear points having rectangular hyperbolas as polar conics entails that the polar conic of any point is a rectangular hyperbola, we only need to find three such points and it is convenient to choose the vertices of ABC. In this case, the equations of those polar conics are : $F'_x = 0, F'_y = 0, F'_z = 0$.

- Hence a $\mathcal{K}_{60}$ is a $\mathcal{K}_{60}^+$ if and only if there exists one point not lying on $\mathcal{L}^\infty$ whose polar conic is a rectangular hyperbola.

- We have seen that a $\mathcal{K}_{60}$ is a $\mathcal{K}_{60}^+$ if and only if the polar lines of the points at infinity of AB, BC, CA concur and is $\mathcal{K}_{60}^{++}$ if and only if they concur on the cubic.

  The common point of the three asymptotes of a $\mathcal{K}_{60}^+$ is the intersection of the diameters of the cubic (see §2.1.2) i.e. the intersection of the polar lines of two points on $\mathcal{L}^\infty$. Once again, it will be convenient to choose two points such as $P_c(1:-1:0)$ and $P_b(1:0:-1)$ on ABC sidelines and solve the following system :

$$\begin{cases} xF'_x(P_c) + yF'_y(P_c) + zF'_z(P_c) = 0 \\ \\ xF'_x(P_b) + yF'_y(P_b) + zF'_z(P_b) = 0 \end{cases}$$

Remark :

Another technique would be to find the center of the poloconic of $\mathcal{L}^\infty$ which is a circle inscribed in the equilateral triangle formed by the asymptotes (see §§2.3.4 and 5.1). Although the computation is often very tedious, this method gives the center of this triangle for any $\mathcal{K}_{60}$ and an equation of the circle.

5.3 Properties of $\mathcal{K}_{60}^+$ cubics

These curves are called "harmonic curves" i.e. curves whose Laplacian identically vanishes. They are sometimes called "stelloïdes" (Lucas), "potential curves" (Coolidge, Kasner), "orthic curves" or "apolar curves" (Brooks), "équilatères" (Serret), "rhizic curves" (Walton). The McCay cubic **K003** is probably the most famous harmonic cubic.

A magnetostatic interpretation is given at the end of this chapter.

The three following chapters are devoted to $\mathcal{K}_{60}^+$ isocubics i.e. $p\mathcal{K}_{60}^+$ and $n\mathcal{K}_{60}^+$ with some further information for unicursal $c\mathcal{K}_{60}^+$.

5.3.1 Barycentric Laplacian of a curve

In the plane of the usual reference triangle ABC with sidelengths a, b, c, we consider a curve of degree n $(n > 0)$ with barycentric equation $F(x, y, z) = 0$. If this same plane is defined by an arbitrary orthonormal cartesian system of coordinates (X, Y), the curve has an equation of the form $G(X, Y) = 0$ and the vertices of ABC are $A(X_A, Y_A)$, $B(X_B, Y_B)$, $C(X_C, Y_C)$.

The correspondence between these two systems of coordinates is given by :

$$x = \begin{vmatrix} X_B - X & X_C - X \\ Y_B - Y & Y_C - Y \end{vmatrix} ; y = \begin{vmatrix} X_C - X & X_A - X \\ Y_C - Y & Y_A - Y \end{vmatrix} ; z = \begin{vmatrix} X_A - X & X_B - X \\ Y_A - Y & Y_B - Y \end{vmatrix} \quad (*)$$

and we shall provisionally write $x = a_1X + a_2Y + a_3$, etc, in order to simplify our calculations.

In this first paragraph, our main concern is to give a barycentric expression of the Laplacian of F.

In cartesian coordinates, the Laplacian of G is defined by

$$\Delta G = \frac{\partial^2 G}{\partial X^2} + \frac{\partial^2 G}{\partial Y^2}.$$

Since

$$\frac{\partial G}{\partial X} = a_1 \frac{\partial F}{\partial x} + b_1 \frac{\partial F}{\partial y} + c_1 \frac{\partial F}{\partial z},$$

we have

$$\frac{\partial^2 G}{\partial X^2} = a_1 U + b_1 V + c_1 W,$$

where

$$U = a_1 \frac{\partial^2 F}{\partial x^2} + b_1 \frac{\partial^2 F}{\partial x \partial y} + c_1 \frac{\partial^2 F}{\partial x \partial z}, \quad V \text{and } W \text{ likewise,}$$

thus

$$\frac{\partial^2 G}{\partial X^2} = \sum_{\text{cyclic}} \left(a_1^2 \, \frac{\partial^2 F}{\partial x^2} + 2b_1 c_1 \, \frac{\partial^2 F}{\partial y \partial z} \right),$$

and similarly

$$\frac{\partial^2 G}{\partial Y^2} = \sum_{\text{cyclic}} \left(a_2^2 \, \frac{\partial^2 F}{\partial x^2} + 2b_2 c_2 \, \frac{\partial^2 F}{\partial y \partial z} \right).$$

It follows that

$$\Delta G = \sum_{\text{cyclic}} (a_1^2 + a_2^2) \, \frac{\partial^2 F}{\partial x^2} + 2(b_1 c_1 + b_2 c_2) \, \frac{\partial^2 F}{\partial y \partial z}.$$

From the equations (*) above we obtain

$$a_1^2 + a_2^2 = (Y_B - Y_C)^2 + (X_C - X_B)^2 = BC^2 = a^2,$$

and

$$\begin{aligned} b_1 c_1 + b_2 c_2 &= (Y_C - Y_A)(Y_A - Y_B) + (X_A - X_C)(X_B - X_A) \\ &= -\overrightarrow{AB}.\overrightarrow{AC} = -\frac{1}{2}(b^2 + c^2 - a^2) = -S_A, \end{aligned}$$

hence finally

$$\Delta G = \sum_{\text{cyclic}} \left(a^2 \, \frac{\partial^2 F}{\partial x^2} - 2 \, S_A \, \frac{\partial^2 F}{\partial y \partial z} \right) = \sum_{\text{cyclic}} S_A \left(\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} - 2 \, \frac{\partial^2 F}{\partial y \partial z} \right),$$

which is called the barycentric Laplacian ΔF of F.

5.3.2 Laplacian of some lower degree curves

1. When the curve is a straight line ($n = 1$), ΔF is obviously null hence a line is always a (trivial) harmonic curve.

2. When the curve is a conic ($n = 2$) with barycentric equation

$$\sum_{\text{cyclic}} \left(\alpha_1 \, x^2 + \beta_1 \, yz \right) = 0,$$

the Laplacian becomes

$$\Delta F = 2 \sum_{\text{cyclic}} \left(a^2 \, \alpha_1 - S_A \, \beta_1 \right).$$

It follows that $\Delta F = 0$ if and only if the conic is a rectangular hyperbola.

In other words, a conic is a harmonic curve if and only if it is a rectangular hyperbola.

3. When the curve is a cubic ($n = 3$), the locus of point P such that $\Delta F(P) = 0$ is generally a line called the "orthic line" of the cubic. This is the locus of point P whose polar conic with respect to the cubic is a rectangular hyperbola. It is also the mixed polar line of the circular points at infinity with respect to the cubic.

   If the polar conic of any point of the plane is a rectangular hyperbola then the cubic is harmonic and it is a $\mathcal{K}_{60}^{+}$.

5.3.3 General harmonic curves or stelloids

A curve $\mathcal{S}_n$ of degree $n > 1$ is said to be a harmonic curve (or a stelloid or an orthic curve) if and only if its Laplacian identically vanishes i.e. $\Delta F(P) = 0$ for any point P.

Recall that a conic is harmonic if and only if it is a rectangular hyperbola.

From [4, 8, 37, 43], we have the following properties :

1. The polar curve of any order of any point with respect to $\mathcal{S}_n$ is also a harmonic curve. In particular, the polar conic of any point P with respect to $\mathcal{S}_n$ is a rectangular hyperbola.

2. $\mathcal{S}_n$ is apolar with respect to the circular points at infinity J_1, J_2. This means that the polar curve of any degree m of J_1 degenerates into m lines passing through J_2 and conversely.

3. $\mathcal{S}_n$ has n real concurring asymptotes and the angle between two consecutive asymptotes is π/n.

4. The point of concurrence X of these n asymptotes is called the "radial center" ("centre de rayonnement" in [27]) of $\mathcal{S}_n$. It is the isobarycenter (or centroid) of the intersections $M_1, M_2, ..., M_n$ of any line through X with the curve. In other words, the sum of the algebraic distances from X to these n points is null. X usually does not lie on $\mathcal{S}_n$.

5. X is also the isobarycenter of all the points on the curve where the tangents have a same arbitrary direction.

6. The hessian of a harmonic curve is a circular curve passing through X. The mutiplicity of each circular points at infinity is $(n - 2)$ when the degree of $\mathcal{S}_n$ is $n > 2$.

7. Any $\mathcal{S}_n$ is the locus of point M such that the sum of the n angles formed by the lines passing through M and n suitably chosen points $M_1, M_2, \ldots, M_n$ on the curve with an arbitrary direction is constant modulo π. These points M_i form a group of n associated pivots and there are infinitely many groups of n associated pivots on the curve.

   If M_1 is a given point on $\mathcal{S}_n$, each isotropic line through M_1 meets $\mathcal{S}_n$ again at $(n - 1)$ points. The $(n - 1)^2$ intersections of a line through J_2 and one point on M_1J_1 with a line through J_1 and one point on M_1J_2 give $(n - 1)$ real points which are the remaining pivots on $\mathcal{S}_n$. The other $(n-1)(n-2)$ intersections are imaginary. See [27].

These curves are in many ways analogous to rectangular hyperbolas and may be seen as rectangular hyperbolas of higher degree.

5.3.4 Isogonal transform of a circum-stelloid

Suppose that $\mathcal{S}_n$ is a circum-stelloid of degree $n > 1$. When A, B, C are not multiple points on the curve, its isogonal transform $\mathcal{S}_n^*$ is a circum-curve of degree $2n - 3$.

Since $\mathcal{S}_n$ meets the line at infinity at n real distinct points, $\mathcal{S}_n^*$ meets the circumcircle $(\mathcal{O})$ at $4n - 6$ points among them the n vertices of a regular polygon and then the remaining $3(n - 2)$ points must be A, B, C each with multiplicity $n - 2$. It follows that $\mathcal{S}_n^*$ must meet each sideline of ABC again at one and only one point (which is therefore always real).

5.3.5 Harmonic cubics or $\mathcal{K}_{60}^+$

A harmonic cubic $\mathcal{S}_3$ is defined by a specific orientation with respect to a group of three pivots (in [27]) or a group of three roots ("racines" in [42]) or a triad on the curve (in [6]).

$\mathcal{S}_3$ has three real concurring asymptotes making $\pi/3 = 60°$ angles with one another and conversely, any cubic with this property is a $\mathcal{S}_3$ (see [27], VI).

The circumcircle of any group of three pivots meets the cubic at three other points which are the vertices of an equilateral triangle.

There are quite many identified $\mathcal{K}_{60}^+$ in [29]. See the annexe at the end. Among them, we find the McCay cubic **K003**, the Kjp cubic **K024**, the third Musselman cubic **K028**, etc. Most of them are circumcubics but this is absolutely not a necessity, see **K077**, **K078**, **K100**, **K258** for instance.

Some of these cubics are $\mathcal{K}_{60}^{++}$ i.e. cubics with asymptotes concurring at X on the curve and thus they are central cubics. This is the case of **K026** ($X = X_5$), **K080** ($X = O$), **K213** ($X = G$), **K525** ($X = H$), etc. Note that the hessian of such cubic is always a central focal cubic with focus X.

When a group of pivots is formed by a double point P and another point Q, the cubic has a node at this double point with perpendicular nodal tangents. See **K028** ($X = X_{381}$) for example. It follows that all the circles passing through P and Q meet the cubic at the vertices of an equilateral triangle.

Obviously, if the three pivots coincide then the cubic decomposes into three lines concurrent at this point and forming a "compass rose" ("rose des vents" or $\mathcal{R}_3$ in [27]).

5.3.6 Harmonic cubics with pivots A, B, C

We illustrate some of the properties above with the circumscribed cubics $\mathcal{S}_3$ having the points A, B, C as a group of associated pivots hence for which the radial center X is G.

Any such cubic is a member of the pencil of cubics generated by the McCay cubic **K003** and the Kjp cubic **K024**. See **Table 22** in [29]. Since this pencil is stable under isogonality, any cubic meets the circumcircle $(\mathcal{O})$ again at the vertices of an equilateral triangle. All these cubics contain A, B, C and six other imaginary points, two by two isogonal conjugate, on the perpendicular bisectors of ABC. These six points are called the "antipoints" of A, B, C by Cayley and Salmon. Two points and two corresponding antipoints may be regarded as the four foci of a same ellipse.

This pencil contains one and only one cubic $\mathcal{S}_3(P)$ passing through a given point P which is not one of the points A, B, C and thus is characterized by a specific orientation θ.

It follows that $\mathcal{S}_3(P)$ is

- the locus of point M such that $(AP, AM) + (BP, BM) + (CP, CM) = 0$ (mod.π),
- equivalently, the locus of point M such that $(AM, L) + (BM, L) + (CM, L) = \theta$ (mod.π), where L is any arbitrary line.

Construction of $\mathcal{S}_3(P)$ when P (or θ) is given

Let $\mathcal{H}(P)$ be the rectangular hyperbola with center the midpoint of BC, passing through B, C, P. This meets the circumcircle $(\mathcal{O})$ of triangle ABC again at two points lying on a line $\mathcal{L}$ passing through O which is parallel to the polar line of O in $\mathcal{H}(P)$.

The orientation θ of the stelloid is then defined by the angle $(\mathcal{L}, AP^*)$ where P^* is the isogonal conjugate of P.

The construction of the cubic is obtained through the rotation about A of two variable lines $\mathcal{D}$, $\mathcal{D}_\theta$ passing through A and making the angle θ. Construct

1. the isogonal transform $\mathcal{D}_\theta^*$ of $\mathcal{D}_\theta$,
2. the parallel at O to $\mathcal{D}$ meeting $(\mathcal{O})$ at M_1, M_2,
3. the rectangular hyperbola passing through B, C, M_1, M_2,
4. its two intersections with $\mathcal{D}_\theta^*$ which are two points on the cubic.

Note that the intersection N of $\mathcal{D}_\theta^*$ and the line M_1M_2 lies on the rectangular hyperbola passing through A, O and the three common points of the cubic with $(\mathcal{O})$. These are the vertices of an equilateral triangle.

Construction of the two pivots P_2, P_3 associated with P on $\mathcal{S}_3(P)$

Let P' be the complement of P and let $\mathcal{E}(P)$ be the ellipse passing through P' which is confocal with the inscribed Steiner ellipse of ABC with real foci F_1, F_2. Draw

1. the tangent at P' to $\mathcal{E}(P)$ i.e. the external bisector of the angle $(P'F_1, P'F_2)$,
2. the two tangents from P to $\mathcal{E}(P)$ intersecting the previous tangent at the requested points P_2, P_3. See figure 5.1.

Properties of $\mathcal{S}_3(P)$

- There are infinitely many groups of pivots on $\mathcal{S}_3(P)$ among them A, B, C. The centroid of any group of pivots is G.
- Any line passing through G meets the cubic $\mathcal{S}_3(P)$ at three points whose centroid is G.
- The circumcircle of any group of pivots meets the cubic again at three points which are the vertices of an equilateral triangle. Thus, any $\mathcal{S}_3(P)$ contains infinitely many inscribed equilateral triangles.

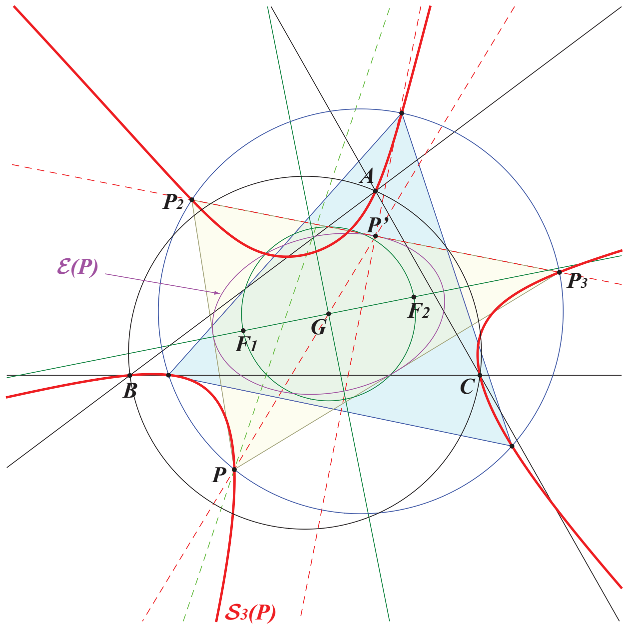

Figure 5.1: Construction of the two pivots P_2, P_3 on $\mathcal{S}_3(P)$

- F_1, F_2 are the fixed points of the involution that carries any point of the plane to the center of its polar conic in the cubic. In other words and following Lucas in [42], these are the *central points* of any group of pivots on $\mathcal{S}_3(P)$.[1] This means that if P_1, P_2, P_3 are three pivots of a same group then

$$\frac{\overrightarrow{P_1P}}{(P_1P)^2}+\frac{\overrightarrow{P_2P}}{(P_2P)^2}+\frac{\overrightarrow{P_3P}}{(P_3P)^2}=\overrightarrow{0} \iff P=F_1 \text{ or } P=F_2$$

- When P is not F_1 nor F_2, let

$$\overrightarrow{E(P)}=\frac{\overrightarrow{P_1P}}{(P_1P)^2}+\frac{\overrightarrow{P_2P}}{(P_2P)^2}+\frac{\overrightarrow{P_3P}}{(P_3P)^2}\neq\overrightarrow{0}.$$

  Following [42], p.6, the line $P, \overrightarrow{E(P)}$ is the tangent at P to $\mathcal{S}_3(P)$.

- Consequently, the inscribed Steiner ellipse of any triangle whose vertices are three pivots of a same group has its real foci at F_1, F_2. Hence these three pivots lie on a same ellipse with center G whose foci are the anticomplements of F_1, F_2.

- There are two nodal $\mathcal{S}_3(P)$ obtained when P is one of two foci F_1, F_2 and then the nodal tangents are obviously perpendicular. Each cubic is the isogonal transform of the other. In this case, the two groups of pivots consist in one focus counted twice and its anticomplement (which is a focus of the Steiner ellipse).

Hessian of $\mathcal{S}_3(P)$

[1] In the complex plane, if the affixes of A, B, C are α, β, γ then the affixes of F_1, F_2 are the roots of the derivative of the polynomial $(z-\alpha)(z-\beta)(z-\gamma)$.

The hessian of any $\mathcal{S}_3(P)$ is a focal cubic with focus G passing through the isodynamic points X_{15} and X_{16}. This hessian meets the line at infinity at a real point whose polar conic in $\mathcal{S}_3$ is the union of the axes of the Steiner inscribed ellipse.

Since the cubics $\mathcal{S}_3(P)$ belong to the pencil generated by **K003** and **K024** their hessians belong to the pencil of focal cubics generated by **K048** and **K193** which also contains **K508**.

Figure 5.2 shows the McCay cubic **K003** with its hessian **K048** and Figure 5.3 shows the Kjp cubic **K024** and with hessian **K193**.

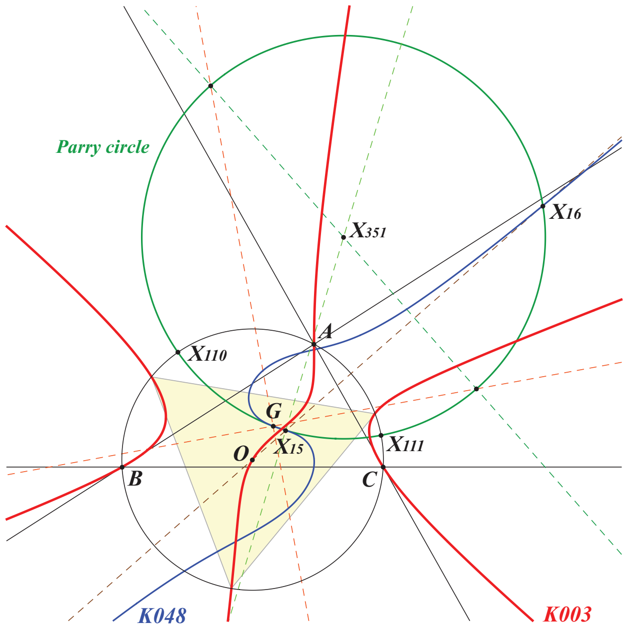

Figure 5.2: The McCay cubic **K003** and its hessian **K048**

All these hessian cubics contain their singular focus G, the isodynamic points X_{15}, X_{16}, the circular points at infinity with tangents passing through G, two imaginary conjugate points which are the common points of $(\mathcal{O})$ and the Lemoine axis, two other remaining points on $(\mathcal{O})$ and on a line passing through X_{23}. Their orthic lines pass through X_{187} hence their real asymptotes pass through the reflection S of G about X_{187}. Each hessian cubic meets its real asymptote again at a point lying on a rectangular hyperbola $\mathcal{H}$ with center X_{187}, with asymptotes parallel to those of the Kiepert hyperbola (or to the axes of the Steiner ellipse), passing through G, X_{15}, X_{16}, X_{98}, X_{99}, X_{385}.

Figure 5.4 shows several hessian cubics of the pencil namely **K048**, **K193** and **K508**.

Naturally, each cubic $\mathcal{S}_3(P)$ which is not unicursal meets its hessian at their nine common inflexion points and three of them are real (and collinear). These inflexion points lie on a bicircular sextic with two real asymptotes parallel to the axes of the Steiner ellipse and passing through the reflection of X_{187} about G. This sextic contains G, X_{15}, X_{16}, the imaginary conjugate points cited above and the four foci of the Steiner inscribed ellipse which are nodes on the curve. It also passes through the common points of $(\mathcal{O})$ and the Thomson cubic **K002**.

Figure 5.5 shows this sextic, the McCay cubic **K003** and its hessian **K048**.

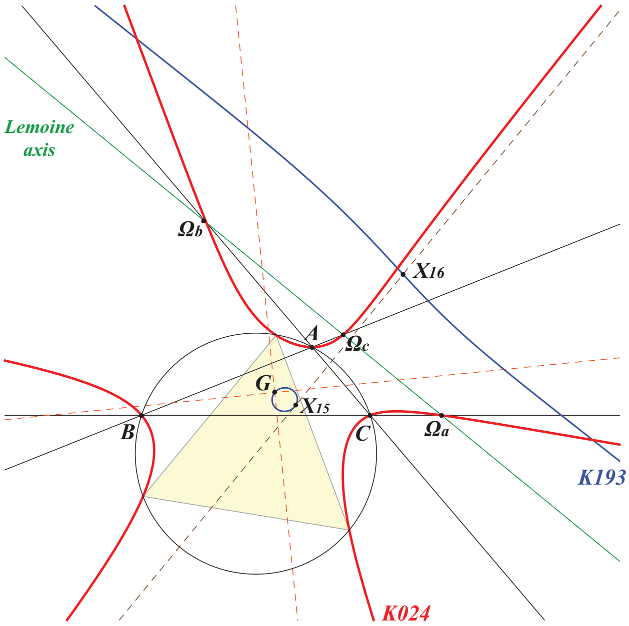

Figure 5.3: The Kjp cubic **K024** and its hessian **K193**

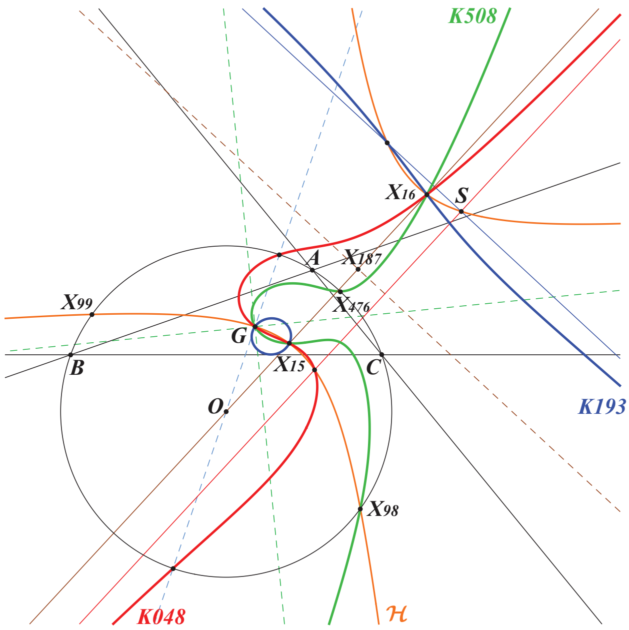

Figure 5.4: Hessian cubics of the stelloids $\mathcal{S}_3(P)$

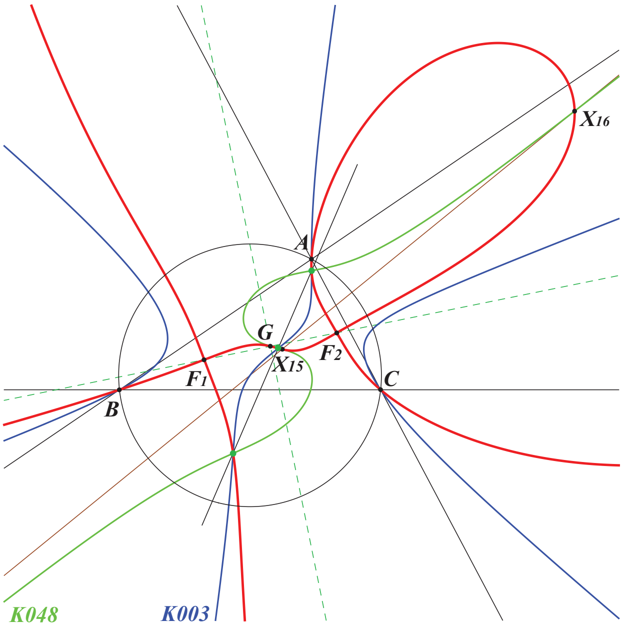

Figure 5.5: The sextic together with the McCay cubic **K003** and its hessian **K048**

5.3.7 Harmonic circumcubics

A harmonic circumcubic is a harmonic cubic passing through A, B, C which is obviously the case when the three pivots of a group are precisely A, B, C.

Suppose that $\mathcal{S}_3$ is a harmonic cubic with group of pivots P_1, P_2, P_3 distinct of A, B, C. This is the locus of point M such that $(P_1M, L)+(P_2M, L)+(P_3M, L) = \theta$ (mod.π), where θ is a given orientation and L any arbitrary line.

According to the previous paragraph, when P_1, P_2, P_3 are the vertices of a proper triangle the cubic $\mathcal{S}_3$ must be a member of the pencil of cubics generated by the McCay cubic and the Kjp cubic of the triangle $P_1P_2P_3$. It immediately follows that there is generally no such cubic passing simultaneously through A, B, C.

Harmonic cubics with pivots H, P, P^*

This configuration always gives a harmonic circumcubic with asymptotes parallel to those of the McCay cubic **K003** and concurring at the centroid X of triangle HPP^*. This cubic always meets the circumcircle $(\mathcal{O})$ at the same points as the isogonal pivotal cubic with pivot the homothetic of X under $h(X_5, 3)$. This pivot lies on the harmonic cubic.

Several examples are **K028** ($P = X_3$, $P^* = X_4$), **K516** ($P = \Omega_1$, $P^* = \Omega_2$, Brocard points), **K358** ($P = F_1$, $P^* = F_2$). See Figure 5.6.

Harmonic nodal circumcubics

Let $P = P_1 = P_2$ be a "double" pivot and let $Q = P_3$ be another point. In general, there is one and only one harmonic circumcubic with node at P.

For example,

– with $P = X_4$ we have $Q = X_3$ and the cubic is **K028**,

– with $P = X_5$ we have $Q = X_{143}$ and the cubic is **K054**, a generalized Lemoine cubic.

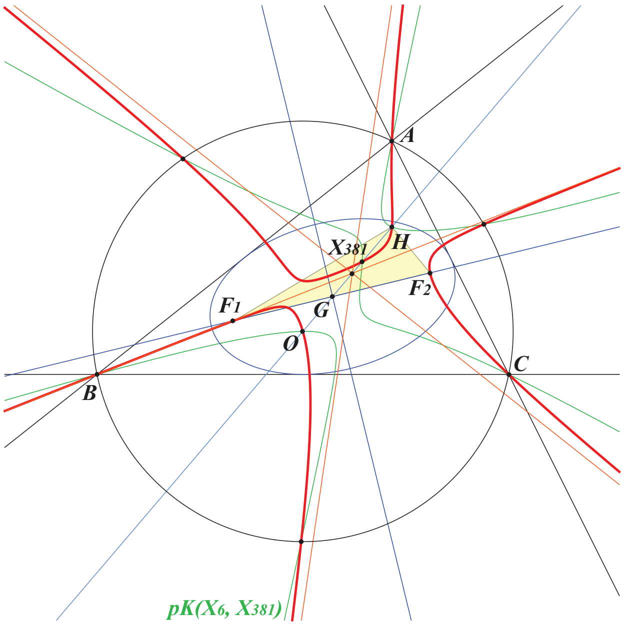

Figure 5.6: The cubic **K358**

Note that the cubic degenerates when P and Q coincide at one of the Fermat points X_{13} and X_{14}.

A special case is obtained when P is X_{80} (or one of its extraversions). Any point Q on the circumcircle gives a harmonic circumcubic with node at X_{80} and all these cubics form a pencil. In particular, when $Q = X_{2222}$ the cubic is **K230**, a conico-pivotal cubic. Naturally, the point of concurrence of the asymptotes of all these cubics lies on the homothetic of the circumcircle under the homothety $h(X_{80}, 1/3)$. Note that all the cubics pass through two fixed points O_1, O_2 on the circumcircle and on a line through X_{900}.

Figure 5.7 shows one of these nodal cubics namely that with asymptotes parallel to those of the McCay cubic **K003**. This contains X_4 and X_{953}.

Harmonic circumcubics with given radial center X

All these harmonic circumcubics form a pencil hence there is one and only one central harmonic circumcubic with center X.

5.3.8 A magnetostatic interpretation

Consider three parallel electric wires (supposed of infinite length) carrying a same electric current (same intensity, same direction). These wires meet a perpendicular plane (containing the reference triangle ABC) at P_1, P_2, P_3 and create a magnetostatic field.

Up to a multiplicative constant, the field at P is of the form

$$\overrightarrow{E(P)} = \frac{\overrightarrow{P_1P}}{(P_1P)^2} + \frac{\overrightarrow{P_2P}}{(P_2P)^2} + \frac{\overrightarrow{P_3P}}{(P_3P)^2}$$

as seen above. It follows that the field lines are precisely our cubics $\mathcal{K}_{60}^{+}$ and the equipotential curves are cassinian curves of degree 6. They are orthogonal to the cubics $\mathcal{K}_{60}^{+}$.

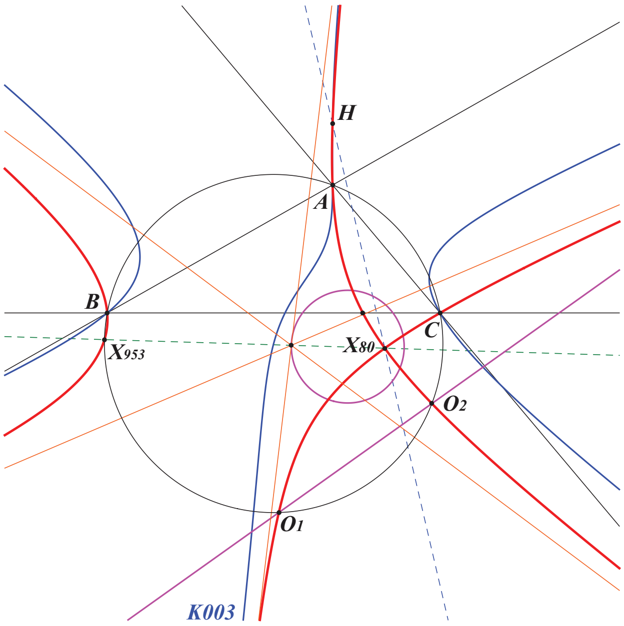

Figure 5.7: A nodal cubic with node X_{80} together with the McCay cubic **K003**

Naturally, it is convenient to choose A, B, C for P_1, P_2, P_3 in which case the field lines are the cubics of the pencil generated by the McCay and Kjp cubics. It is then (physically) clear that any two lines cannot meet at other points than A, B, C and the angles made by these two lines at A, B, C must be the same. In particular, the McCay and Kjp cubics are orthogonal at these points. See Figure 5.8 and Figure 5.9.

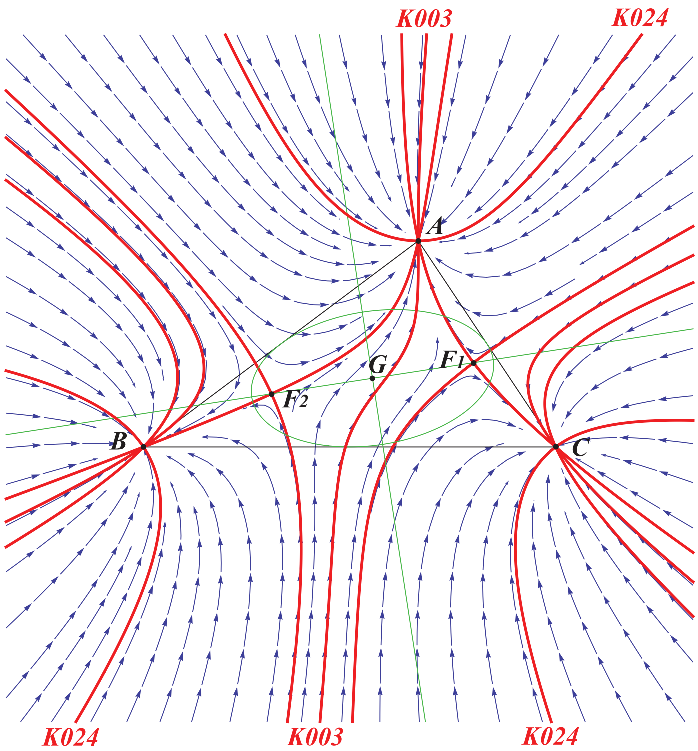

Figure 5.8: McCay cubic **K003**, Kjp cubic **K024** and field lines

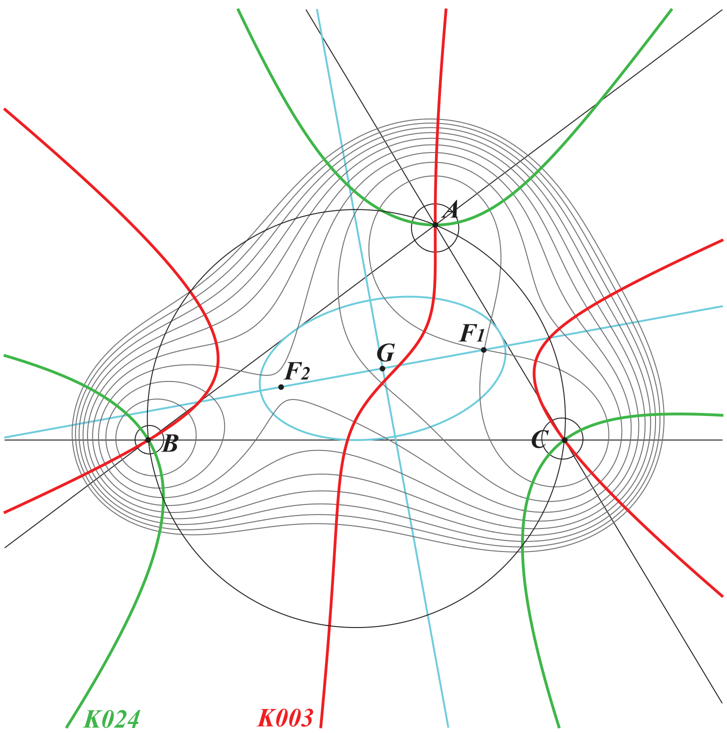

Figure 5.9: McCay cubic **K003**, Kjp cubic **K024** and cassinian curves

5.3.9 Annexe : list of harmonic cubics

The following tables gives a selection of stelloids or $\mathcal{K}_{60}^{+}$.

Notations :

X : intersection of the asymptotes

o : the cubic is a circumcubic

c : the cubic is a central cubic or a $\mathcal{K}_{60}^{++}$ whose center is X

n : the cubic is a nodal cubic

the **red cubics** are those with asymptotes parallel to those of the McCay cubic **K003** i.e. perpendicular to the sidelines of the Morley triangle

the **green cubics** are those with asymptotes parallel to those of the Kjp cubic **K024** i.e. parallel to the sidelines of the Morley triangle

Notes : in [29],

1. **Table 22** gives cubics of the pencil generated by the McCay cubic **K003** and the Kjp cubic **K024**. These are the harmonic cubics having A, B, C as a group of pivots hence their asymptotes concur at G.
2. **CL006** is the class of pivotal harmonic cubics or $p\mathcal{K}_{60}^{+}$.
3. The pseudo-pivotal cubics $ps\mathcal{K}$ are defined and studied in [30].
4. The sympivotal cubics $sp\mathcal{K}$ are described in **CL055** and **CL056**.

Table 5.1: Stelloids or $\mathcal{K}_{60}^{+}$ (part 1)

cubic	X	o	c	n	type	centers on the curve
K003	X_2	o			$p\mathcal{K}$	X_1, X_3, X_4, X_{1075}, X_{1745}, X_{3362}
K024	X_2	o			$n\mathcal{K}_0$	none
K026	X_5	o	c		$ps\mathcal{K}$	X_3, X_4, X_5
K028	X_{381}	o		n	$ps\mathcal{K}$	X_3, X_4, X_8, X_{76}, X_{847}
K037	X_{476}	o			$p\mathcal{K}$	X_{30}
K046a	X_{618}	o	c		$p\mathcal{K}$	X_{13}, X_{616}, X_{618}
K046b	X_{619}	o	c		$p\mathcal{K}$	X_{14}, X_{617}, X_{619}
K049	X_{51}	o			$p\mathcal{K}$	X_4, X_5, X_{52}, X_{847}
K054	midpoint of X_5, X_{51}	o		n		X_4, X_5, X_{143}
K071	reflection of X_{51} in X_5	o			$ps\mathcal{K}$	X_4, X_5, X_{20}, X_{76}
K077	X_{376}					X_1, X_3, X_{20}, X_{170}, X_{194}
K078	$\overrightarrow{OX} = 1/9\ \overrightarrow{OH}$					X_1, X_2, X_3, X_{165}
K080	X_3	o	c			X_3, X_4, X_{20}, X_{1670}, X_{1671}
K094	X_{599}	o			$n\mathcal{K}$	none
K097	$X_{79} \times X_{1654}$	o			$p\mathcal{K}$	X_1, X_{79}
K100	X_3		c			X_1, X_3, X_{40}, X_{1670}, X_{1671}
K115	$X_2X_{511} \cap X_6X_{22}$	o			$\mathcal{K}_0$	X_4
K139	$X_{1989} \div X_{30}$	o	c		$n\mathcal{K}$	X_{30}
K205	$X_2X_{249} \cap X_{111}X_{265}$	o	c		$n\mathcal{K}$	none
K213	X_2	o	c		$n\mathcal{K}$	X_2
K230	E_{597}	o		n	$c\mathcal{K}$	X_{80}, X_{2222}
K258	X_{549}			n		X_1, X_3, X_5, X_{39}, X_{2140}
K268	$X_2X_{511} \cap X_3X_{64}$	o			$\mathcal{K}_0$	X_4, X_{20}, X_{140}
K309	$\overrightarrow{OX} = 2/9\ \overrightarrow{OH}$	o				X_3, X_4, X_{376}, X_{1340}, X_{1341}
K358	$\overrightarrow{OX} = 5/9\ \overrightarrow{OH}$	o				X_3, X_4, X_{381}
K412	$\overrightarrow{X_5X} = 1/3\ \overrightarrow{X_5X_{51}}$	o				X_2, X_4, X_5, X_{51}, X_{262}
K513	$\overrightarrow{X_{187}X} = 1/3\ \overrightarrow{X_{187}X_{265}}$	o				X_6, X_{15}, X_{16}, X_{74}, X_{265}, X_{3016}
K514	$X_3X_{1506} \cap X_{20}X_{32}$	o				X_4, X_{15}, X_{16}, X_{39}
K515	X_{3258}	o			$p\mathcal{K}$	X_{30}, X_{1138}
K516	X_{262}	o			$\mathcal{K}_0$	X_4, X_{3095}
K525	X_4	o	c		$sp\mathcal{K}$	X_3, X_4, X_{382}
K543	$\overrightarrow{X_{107}X} = 1/3\ \overrightarrow{X_{107}X_{125}}$	o			$p\mathcal{K}$	none
K580	X_{568}	o				X_4, X_{847}
K581	$\overrightarrow{OX} = 4/9\ \overrightarrow{OH}$	o				X_2, X_3, X_4, X_{262}
K582	$\overrightarrow{KX} = 1/3\ \overrightarrow{KX_{381}}$	o				X_2, X_4, X_6, X_{262}
K594	$\overrightarrow{IX} = 1/3\ \overrightarrow{IH}$	o		n		X_1, X_4, X_{1482}
K595	$\overrightarrow{X_{98}X} = 1/3\ \overrightarrow{X_{98}X_{265}}$	o		n		X_{74}, X_{98}, X_{265}, X_{290}, X_{671}
K596	$\overrightarrow{X_{99}X} = 1/3\ \overrightarrow{X_{99}X_{265}}$	o		n		X_{74}, X_{99}, X_{265}, X_{290}
K597	$\overrightarrow{X_{477}X} = 1/3\ \overrightarrow{X_{477}X_{265}}$	o		n		X_{30}, X_{74}, X_{265}, X_{477}
K598	X_6					none

Table 5.2: Stelloids or $\mathcal{K}_{60}^{+}$ (part 2)

cubic	X	o	c	n	type	centers on the curve
K607	$X_{1511}*$	o	c		$n\mathcal{K}$	X_{30}, X_{1511}^{*}
K613	$\overrightarrow{X_5X} = 1/3\ \overrightarrow{X_5X_{110}}$	o			$n\mathcal{K}$	X_4, X_{110}, X_{1113}, X_{1114}
K643	centroid of OHK	o			$sp\mathcal{K}$	X_4, X_6, X_{4846}
K665	X_{549}	o				X_3, X_4, X_{39}, X_{550}
K669	$\overrightarrow{X_5X} = 1/3\ \overrightarrow{X_5X_{265}}$	o			$n\mathcal{K}$	X_4, X_{74}, X_{265}
K670	$\overrightarrow{X_5X} = 1/3\ \overrightarrow{X_5X_{68}}$	o			$ps\mathcal{K}$	X_4, X_{26}, X_{64}, X_{68}, X_{847}
K708	$\overrightarrow{X_4X} = 2/3\ \overrightarrow{X_4X_6}$	o			$sp\mathcal{K}$	X_4, X_{1344}, X_{1345}, X_{1351}
K714	$X_3X_{107} \cap X_5X_{53}$	o			$sp\mathcal{K}$	X_4
K724	on line X_3X_{125}	o		n	$ps\mathcal{K}$	X_{74}, X_{265}, X_{5961}, X_{6344}

Chapter 6

$p\mathcal{K}_{60}$ isocubics

6.1 A crucial point and a crucial cubic

- Let us denote by P_o the point with barycentric coordinates

$$(p_o : q_o : r_o) = \left(\frac{1}{4S_A^2 - b^2c^2} : \frac{1}{4S_B^2 - c^2a^2} : \frac{1}{4S_C^2 - a^2b^2}\right)$$

  P_o is the barycentric product of the Fermat points i.e. the pole of the isoconjugation which swaps them. Also we have $P_o = \mathbf{g}X_{323}$. [1] It is the intersection of the line through the Fermat points X_{13} and X_{14} and the parallel to the Euler line at $X_{50} = \mathbf{gt}X_{323}$. It lies on the circum-hyperbola through G and K and on the rectangular circum-hyperbola through the midpoint of GH [2].

- Let **K095** $= \mathcal{C}_o$ be the pivotal circum-cubic with pivot P_o which is invariant under the conjugation

$$\varphi_o \;: M(x:y:z) \mapsto M^*\left(\frac{a^2p_o}{x} : \frac{b^2q_o}{y} : \frac{c^2r_o}{z}\right) \sim (a^2p_oyz : b^2q_ozx : c^2r_oxy)$$

  which swaps P_o and K. See figure 6.1.

  Its equation is

$$\sum_{\text{cyclic}} (4S_A^2 - b^2c^2)x^2(b^2z - c^2y) = 0.$$

  It passes through P_o, K, X_{53} (symmedian point of the orthic triangle), X_{395}, X_{396}, X_{2160}, X_{2161}, the common point M_o of the orthic axis and the line HK [3], P_o/K (a point on OK which is the tangential of K in $\mathcal{C}_o$).

  The polar conic of K passes through A, B, C, G, K, P_o which shows $\mathcal{C}_o$ is tangent to the symmedians at the vertices of ABC.

6.2 Main theorem for $p\mathcal{K}_{60}$

Theorem :

For a given $\Omega(p:q:r)-$isoconjugation, there is in general one and only one $p\mathcal{K}_{60}$. Moreover, this $p\mathcal{K}_{60}$ is a $p\mathcal{K}_{60}^+$ if and only if Ω lies on $\mathcal{C}_o$.

[1] P_o is now X_{1989} in [39].
[2] This is X_{381} in [38, 39].
[3] This point is $\left[\left(a^2(b^2+c^2-2a^2)+(b^2-c^2)^2\right)/S_A\right]$. It is now X_{1990} in [39].

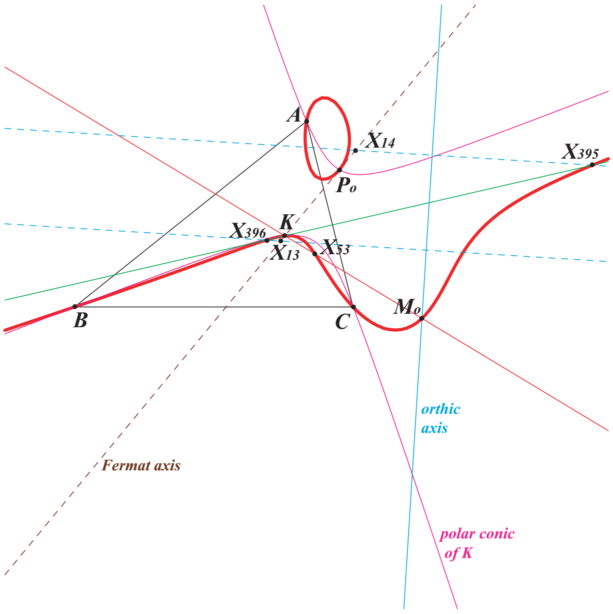

Figure 6.1: The cubic $\mathcal{C}_o$ or **K095**

The pivot is $P(u : v : w)$ where

$$u = (4S_A^2 - b^2c^2)p(q + r - p) - 3(b^4rp + c^4pq - a^4qr),$$

the other coordinates v, w similarly.

Proof :

Starting from equation (1.1) in §1.3, theorem 1 in §5.1 used with two points on $\mathcal{L}^\infty$ such as $P_c(1 : -1 : 0)$ and $P_b(1 : 0 : -1)$ gives two linear equations in u, v, w. This system has in general one solution that gives the pivot $P(u : v : w)$ as above.

u, v, w being replaced in (1.1), theorem 2 in §5.2 shows that the cubic is a $p\mathcal{K}_{60}^+$ if and only if Ω lies on $\mathcal{C}_o$.

A special case :

The system above does not have a unique solution if and only if the pole Ω is one of the three points Ω_a, Ω_b, Ω_c that are the only points for which there are infinitely many $p\mathcal{K}_{60}$ with pole one of these points. They lie on a large number of curves (among them $\mathcal{C}_o$) although they are not constructible with ruler and compass only. In particular, they are the common points, apart $Z = X_5X_{1989} \cap X_6X_{30}$, of the two conics passing through Z and X_5, X_{53}, X_{216}, X_{1989} for the former, X_{395}, X_{396}, X_{523}, X_{3003} for the latter.

6.3 Corollaries

6.3.1 Corollary 1

Let us denote by Φ the mapping $\Omega(p : q : r) \mapsto P(u : v : w)$ and by Ψ the mapping $P \mapsto \Omega$.

Φ has three singular points Ω_a, Ω_b, Ω_c and Ψ has also three singular points U_a, U_b, U_c on the Neuberg cubic and on many other curves. Hence, there are infinitely many $p\mathcal{K}_{60}$ with pivot one of the points U_a, U_b, U_c. See §6.5.1 for more details.

We have :

$$\begin{aligned} p &= \left[a^2(a^2+b^2+c^2) - 2(b^2-c^2)^2\right] u^2 + 3a^4vw \\ &+ \left[a^2(2a^2+2c^2-b^2) - (b^2-c^2)^2\right] uv \\ &+ \left[a^2(2a^2+2b^2-c^2) - (b^2-c^2)^2\right] uw, \\ \\ &= 16\Delta^2 u(2u+v+w) \\ &+ 3a^2[2u(-S_Au+S_Bv+S_Cw) - (-a^2vw+b^2wu+c^2uv)]. \end{aligned}$$

the other two coordinates q, r similarly.

The second form clearly shows that Ω lies on the line passing through the pole of the circular pivotal cubic with pivot P [4] and through the barycentric product of P and **cc**P.

In this case, the following statements are equivalent :

- $p\mathcal{K}$ is a $p\mathcal{K}_{60}$ with pole Ω and pivot P.
- $P = \Phi(\Omega)$.
- $\Omega = \Psi(P)$.

6.3.2 Corollary 2

Ψ maps the Neuberg cubic to the cubic $\mathcal{C}_o$ i.e. any point P on the Neuberg cubic is the pivot of a unique $p\mathcal{K}^+_{60}$ with the pole Ω chosen according to Corollary 1.

In other words, the following statements are equivalent :

- $p\mathcal{K}$ is a $p\mathcal{K}^+_{60}$ with pole Ω and pivot P.
- $\Omega \in \mathcal{C}_o$.
- P is on the Neuberg cubic.

For example, the choice of O for P leads to $\Omega = K$ and we find the famous McCay cubic. See [9] for details.

6.4 Constructions

6.4.1 Construction of P for a given $\Omega \neq K$

We have seen in §4.2.1 that the pivot of a non-isogonal circular pivotal isocubic is

$$P_\Omega = \left[b^2c^2p(q+r-p) - (b^4rp + c^4pq - a^4qr)\right],$$

this point not defined if and only if $\Omega = K$.

Recall that P_Ω is the reflection of S_Ω (tripole of the line ΩK on the circumcircle of ABC) about δ_Ω (trilinear polar of the Ω-isoconjugate of K).

Recall also (see §4.2.2) that the center of $\mathcal{C}_\infty$ is $c_\Omega = [p(q+r-p)]$, the $G-$Ceva conjugate of P.

[4] Recall that this pole is the barycentric product of P and **ig**P.

Lemma 1 :

P, P_Ω, c_Ω are collinear. (this is obvious)

Lemma 2 :

Let us denote by Ω_1 the isotomic conjugate of the P_o-isoconjugate of c_Ω and let Ω_2 = $\mathbf{agtg}\Omega$. Their coordinates are :

$$\Omega_1 = \left[(4S_A^2 - b^2c^2)p(q+r-p)\right]$$

and

$$\Omega_2 = \left[b^4rp + c^4pq - a^4qr\right]$$

P, Ω_1, Ω_2 are collinear. (this is obvious too)

Since we know two lines containing P, the construction is possible although rather complicated.

6.4.2 Construction of Ω when P is on the Neuberg cubic

Let us recall that $P_o = X_{1989}$ is the barycentric product of the Fermat points, that $M_o = X_{1990}$ is the intersection of the orthic axis and the line HK and that a $p\mathcal{K}_{60}$ with pivot on the Neuberg cubic is always a $p\mathcal{K}_{60}^{+}$.

The rectangular circum-hyperbola $\mathcal{H}_1$ through M_o intersects the parallel at P to the Euler line at X_{30} and U_1. [5]

The line HP intersects again the rectangular circum-hyperbola $\mathcal{H}_2$ through P_o (and the midpoint X_{381} of GH) at U_2. [6]

Then $\Omega = M_oU_1 \cap P_oU_2$. See figure 6.2.

In this case, the isopivot P^* is a point of the cubic **K060**, the inverse of the Neuberg cubic in the circumcircle.

An alternative construction

The transformation f we met in §3.5.3 maps P on the Neuberg cubic **K001** onto Ω which lies on **K095**.

6.4.3 Construction of Ω when P is not on the Neuberg cubic

We first recall that Ω lies on the line passing through the two following points :

– the pole Ω_c of the circular pivotal cubic with pivot P, barycentric product of P and $\mathbf{ig}P$,

– the barycentric product Ω_p of P and $\mathbf{cc}P$, pole of a $p\mathcal{K}^+$ with asymptotes concurring at the barycentric product of $\mathbf{cc}P$ and $\mathbf{ct}P$.

We wish to find another line that contains Ω and we will make use of the facts that $\Psi : P \to \Omega$ maps any point Q on the circle $\mathcal{C}(X_{20}, 2R)$ to a point Y on the orthic axis of ABC and that Ψ transforms the rectangular hyperbola $\mathcal{H}(P)$ passing through O, U_a, U_b, U_c and P into a line passing through the Lemoine point K of ABC. See §6.5.1 below for further details about these points.

[5] The asymptotes of this rectangular circum-hyperbola $\mathcal{H}_1$ are perpendicular and parallel to the Euler line. It is the isogonal conjugate of the line through O, X_{74}, X_{110}.

[6] This rectangular circum-hyperbola $\mathcal{H}_2$ is the isogonal conjugate of the line through O, X_{323}.

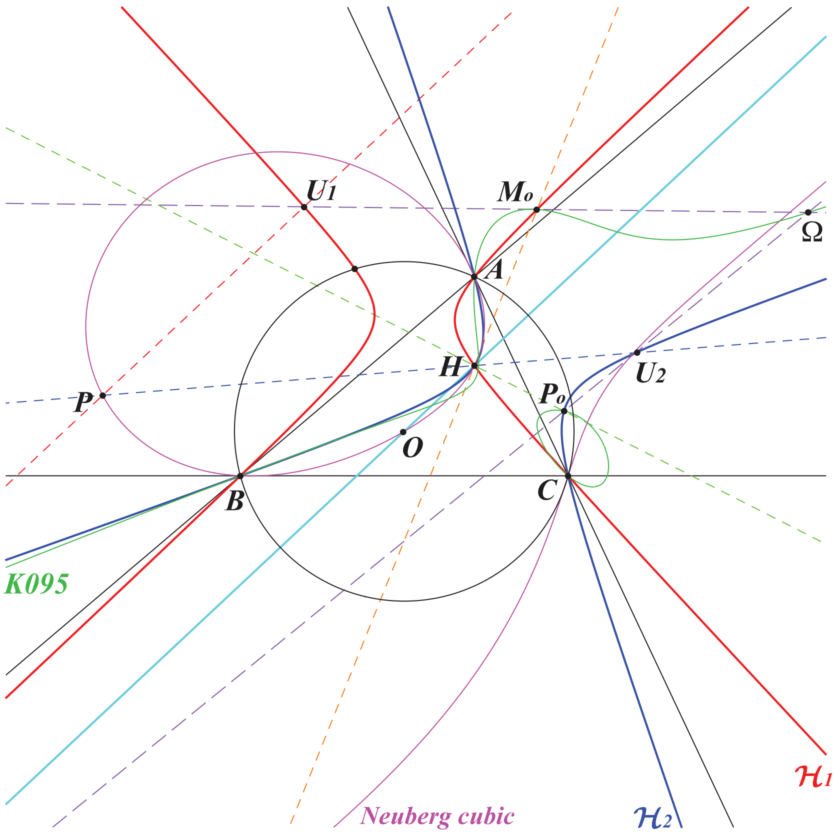

Figure 6.2: Construction of Ω when P is on the Neuberg cubic

- We first need to construct the center W of $\mathcal{H}(P)$ but, since these points U_a, U_b, U_c are not ruler and compass constructable, we use the pencil of rectangular hyperbolas $\mathcal{F}$ passing through O, U_a, U_b, U_c that obviously contains $\mathcal{H}(P)$ and that is generated by the two simple hyperbolas $\mathcal{H}_1$ and $\mathcal{H}_2$:

  – $\mathcal{H}_1$ contains O, X_{20}, X_{399}, X_{1147} with asymptotes parallel to those of the Jerabek hyperbola, with center the image of X_{110} in the translation that maps H to the nine point center X_5,

  – $\mathcal{H}_2$ contains O, X_{616}, X_{617} with asymptotes parallel to those of the Kiepert hyperbola, with center the image of X_{99} in the same translation. See figure 6.3.

  If P is a point not lying on these two hyperbolas, the polar lines of P in $\mathcal{H}_1$, $\mathcal{H}_2$ meet at P' lying on the tangent at P to $\mathcal{H}(P)$.

  The conjugated diameters of the line PP' with respect to $\mathcal{H}_1$, $\mathcal{H}_2$ pass through the centers of $\mathcal{H}_1$, $\mathcal{H}_2$ and meet at T on the nine point circle of $\mathcal{F}$ namely the circle with radius R and center the midpoint X_{550} of O, X_{20}. The second intersection of the line TP with this latter circle is the center W of $\mathcal{H}(P)$. The reflection of O about W is the fourth point Q where $\mathcal{H}(P)$ meets $\mathcal{C}(X_{20}, 2R)$, the three other points being U_a, U_b, U_c. See figure 6.4.

- Let now S be the midpoint of HQ (on the circumcircle $(\mathcal{O})$ of ABC). The isogonal conjugate of the trilinear pole of the perpendicular at O to the Simson line of S is the requested point $Y = \Psi(Q)$ on the orthic axis. This completes the construction

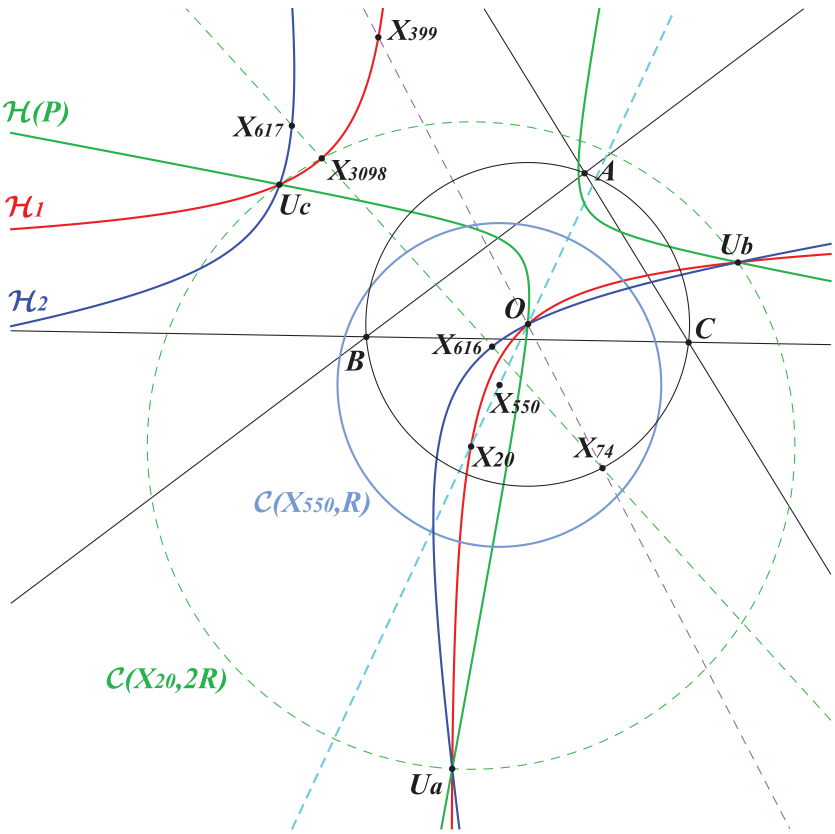

Figure 6.3: Pencil of hyperbolas $\mathcal{H}(P)$

of Ω and thus that of the cubic.

Remark : $\mathcal{H}(P)$ meets the Neuberg cubic **K001** at O, U_a, U_b, U_c and two other (real or not) points P_1, P_2 collinear with X_{74} since X_{74} is the coresidual of O, U_a, U_b, U_c in this cubic. It follows that P_1, P_2 are $X_{30}-$Ceva conjugated thus their midpoint lies on $(\mathcal{O})$ and their perpendicular bisector pass through X_{110}.

6.5 A very special pencil of cubics

6.5.1 The points U_a, U_b, U_c

- Let us call G_a, G_b, G_c the vertices of the anticomplementary (or antimedial) triangle.

  $\mathcal{H}_a$ is the hyperbola through B, C, G_a, the reflection A' of A about the sideline BC, the reflection A_h of H about the second intersection of the altitude AH with the circumcircle. The angle of each of its asymptotes with BC is $60°$. $\mathcal{H}_b$ and $\mathcal{H}_c$ are defined similarly.

  We can note that $\mathcal{H}_a, \mathcal{H}_b$ and $\mathcal{H}_c$ are the images under Φ of the sidelines of triangle ABC.

  $\mathcal{H}_a, \mathcal{H}_b$ and $\mathcal{H}_c$ have three points in common U_a, U_b, U_c lying on the circle centered at L with radius $2R$, passing through A_h, B_h, C_h. The orthocenter of triangle $U_aU_bU_c$ is O and its centroid is a point on the Euler line of triangle ABC.

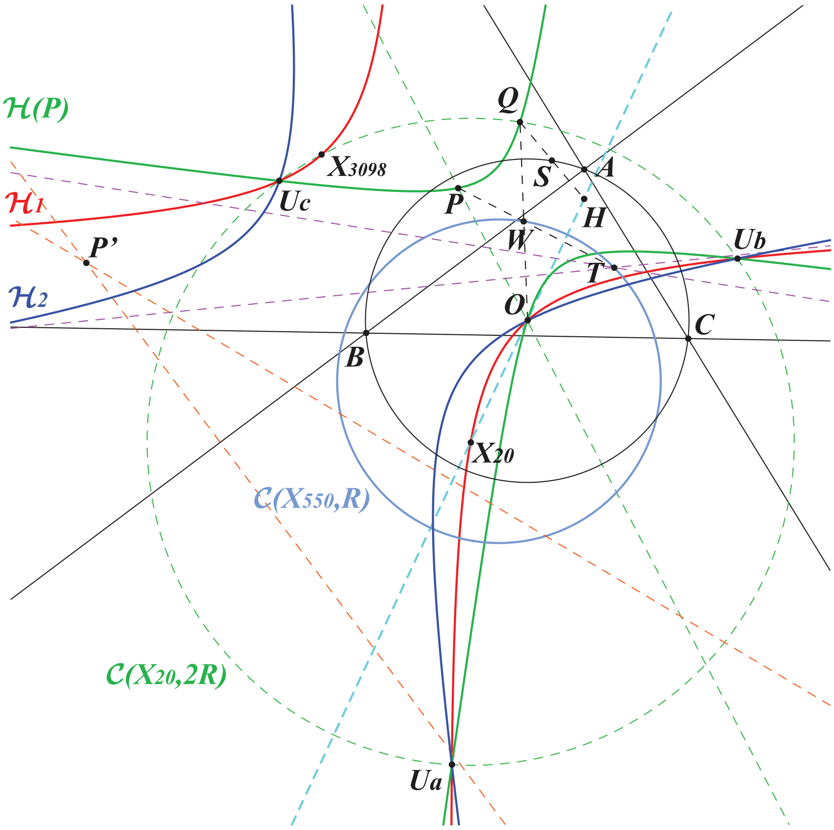

Figure 6.4: Construction of $\mathcal{H}(P) \cap \mathcal{C}(X_{20}, 2R)$

- We remark that the points U_a, U_b, U_c also lie on :
  - the Neuberg cubic.

    Their isogonal conjugates $\mathbf{g}U_a$, $\mathbf{g}U_b$, $\mathbf{g}U_c$, the midpoints V_a, V_b, V_c of triangle $\mathbf{g}U_a\mathbf{g}U_b\mathbf{g}U_c$ which are the complements of $\mathbf{g}U_a$, $\mathbf{g}U_b$, $\mathbf{g}U_c$ and their isogonal conjugates $\mathbf{g}V_a$, $\mathbf{g}V_b$, $\mathbf{g}V_c$ also lie on the Neuberg cubic. This means that we know the 9 points of the Neuberg cubic having their anticomplement on the Neuberg cubic : $O, X_{13}, X_{14}, X_{30}, V_a, V_b, V_c$ and the circular points at infinity. The points $\mathbf{g}U_a$, $\mathbf{g}U_b$, $\mathbf{g}U_c$ lie on the circle $(O, 2R)$ passing through X_{399} and the triangles ABC and $\mathbf{g}U_a\mathbf{g}U_b\mathbf{g}U_c$ share the same orthocenter, circumcenter and centroid.)
  - the only isotomic $p\mathcal{K}_{60}$ (see §6.6.2).
  - the rectangular hyperbola $\mathcal{H}_L$ through $O, L = X_{20}, X_{399}$, the reflection of H about X_{110}, having its asymptotes parallel to those of the Jerabek hyperbola.
  - the rectangular hyperbola $\mathcal{H}_H$ through O, H, the reflection of H about X_{107} (a point on the Neuberg cubic), having its asymptotes parallel to those of the rectangular circum-hyperbola with center X_{122}.

More generally, we remark that all the rectangular hyperbolas of the pencil of conics passing through U_a, U_b, U_c and O are centered on the circle with center X_{550} and radius R.

6.5.2 A theorem

Now we consider the pencil $\mathcal{F}$ of cubics through the points A, B, C, G_a, G_b, G_c, U_a, U_b, U_c.

For instance, the union of $\mathcal{H}_a$ and the line G_bG_c is a degenerate cubic of $\mathcal{F}$.

We have the following theorem :

1. Each cubic of $\mathcal{F}$ is a $\mathcal{K}_{60}$.
2. The locus of the pivots of all $p\mathcal{K}_{60}$ with given asymptotic directions is the member of $\mathcal{F}$ which has the same asymptotic directions.
3. There is only one $\mathcal{K}_{60}^{+}$ in $\mathcal{F}$ and, in fact, it is a $\mathcal{K}_{60}^{++}$ denoted by $\mathcal{K}_{O}^{++}$ or **K080**. Its center is O and the asymptotes are perpendicular to the sides of the (first) Morley triangle. See figure 6.5.

   Its equation is :

$$\sum_{\text{cyclic}} c^2xy\left[\left(a^2S_C + b^2(c^2-b^2)\right)x - \left(b^2S_C + a^2(c^2-a^2)\right)y\right]$$
$$+(a^2-b^2)(b^2-c^2)(c^2-a^2)\,xyz = 0$$

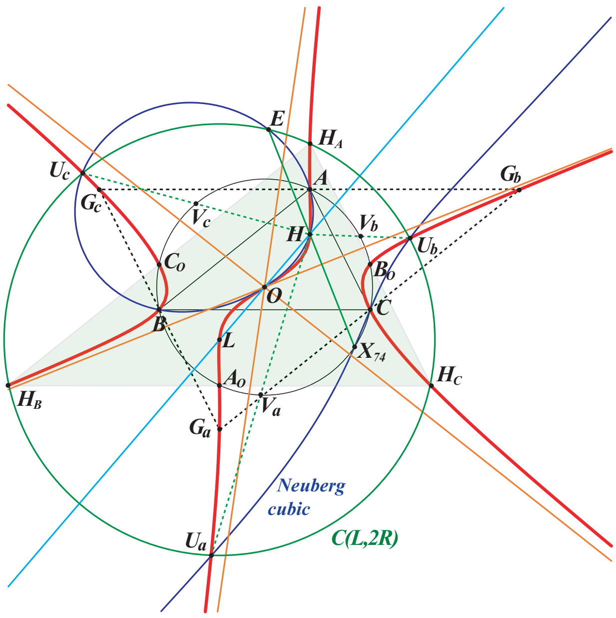

Figure 6.5: **K080** or $\mathcal{K}_{O}^{++}$ and the Neuberg cubic

$\mathcal{K}_{O}^{++}$ passes through O, H, L, the symmetrics H_A, H_B, H_C of H about A, B, C, the symmetrics A_O, B_O, C_O of A, B, C about O. The nine common points of $\mathcal{K}_{O}^{++}$ and the Neuberg cubic are $A, B, C, H, U_a, U_b, U_c$ and O which is double.

6.5.3 K026 or $\mathcal{K}_{N}^{++}$ the first Musselman cubic

The homothety $h_{H,1/2}$ transforms $\mathcal{K}_{O}^{++}$ into another $\mathcal{K}_{60}^{++}$ denoted by $\mathcal{K}_{N}^{++}$ or **K026**. Its center and inflexion point is $N = X_5$ with an inflexional tangent passing through X_{51} (centroid of the orthic triangle). Its equation is :

$$2(a^2-b^2)(b^2-c^2)(c^2-a^2)\,xyz\;+\sum_{\text{cyclic}} a^2\left[(b^2-c^2)^2-a^2(b^2+c^2)\right] yz(y-z)=0$$

in which we recognize the coordinates $\left[a^2[(b^2-c^2)^2-a^2(b^2+c^2)]\right]$ of X_{51}.

The following points lie on it : O, H, the midpoints of ABC, of AH, BH, CH, of HG_a, HG_b, HG_c, of HU_a, HU_b, HU_c (those three on the Napoleon cubic). The tangents at A, B, C pass through X_{51} too.

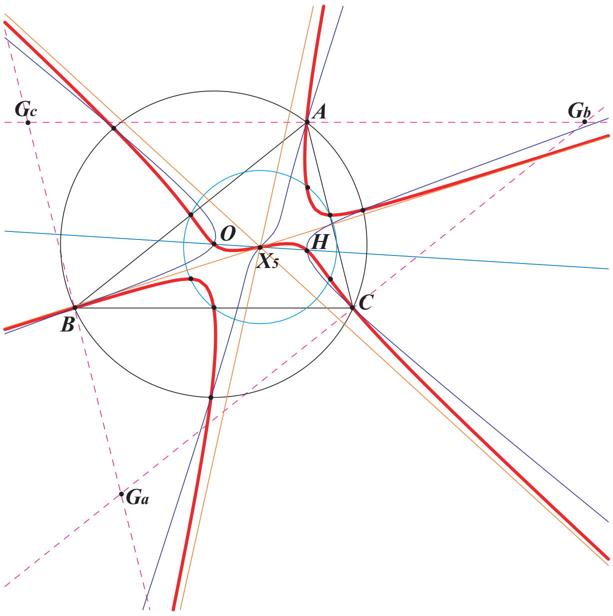

Figure 6.6: **K026** the first Musselman cubic

We know its six common points with the circumcircle, its six common points with the nine-point circle, its nine common points with the Napoleon cubic.

This cubic is mentionned in [46] p.357 : if we denote by B_1, B_2, B_3 the reflections of A, B, C about a point P, by C_1, C_2, C_3 the reflections of P about the sidelines of triangle ABC, by D the common point of circles $AB_2B_3, BB_3B_1, CB_1B_2$ and by E the common point of circles $AC_2C_3, BC_3C_1, CC_1C_2$ then D and E (points on the circumcircle) will coincide if and only if P lies on $\mathcal{K}_N^{++}$.

$\mathcal{K}_N^{++}$ can also be seen as the locus of centers of central isocubics whose asymptotes are perpendicular to the sidelines of the first Morley triangle.

6.6 Some unusual pivotal cubics

6.6.1 The McCay family

The theorem seen above in §6.5 shows that each point on $\mathcal{K}_O^{++}$ is the pivot of a $p\mathcal{K}_{60}$ having the same asymptotic directions than the McCay cubic which is obtained for the points O as pivot and K as pole.

Since H is on $\mathcal{K}_O^{++}$ and on the Neuberg cubic too, there is another remarkable $p\mathcal{K}_{60}^{+}$: its pivot is H, its pole is X_{53} (Lemoine point of the orthic triangle) and the asymptotes concur at X_{51} (centroid of the orthic triangle). This cubic is **K049**, the McCay cubic for the orthic triangle. See figure 6.7. Its equation is :

$$\sum_{\text{cyclic}} \left[(b^2 - c^2)^2 - a^2(b^2 + c^2)\right] \, yz\,(S_B y - S_C z) = 0$$

It is clear that the three points U_a, U_b, U_c are the pivots of three more $p\mathcal{K}_{60}^{+}$ denoted by $p\mathcal{K}_a^{+}$, $p\mathcal{K}_b^{+}$ and $p\mathcal{K}_c^{+}$.

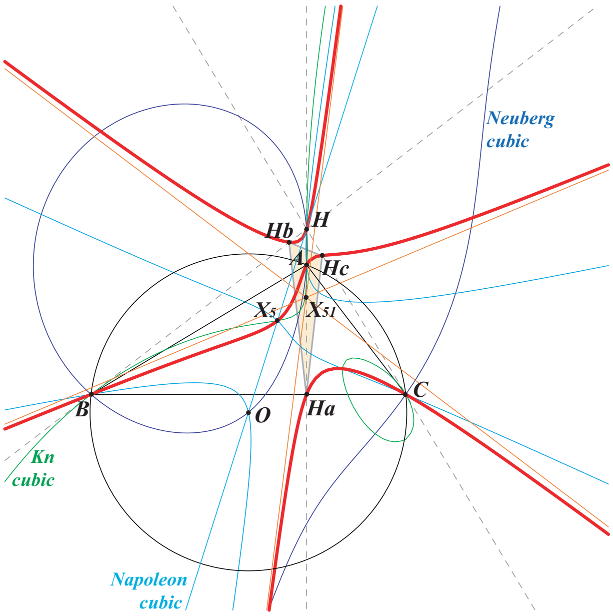

Figure 6.7: **K049** the McCay orthic cubic

Another notable $p\mathcal{K}_{60}$ is obtained when the pivot is L and the pole X_{216}. This cubic **K096** passes through O and X_5 (nine-point center).

6.6.2 **K092** the isotomic $p\mathcal{K}_{60}$

There is one and only one isotomic $p\mathcal{K}_{60}$: it is denoted by $p\mathcal{K}_t$. See figure 6.8.

Its pivot is $P_t = [2(b^4 + c^4 - 2a^4) + 2a^2(b^2 + c^2) - b^2c^2]$ [7]

The isotomic conjugate of P_t is one of the fixed points of the quadratic transformations Φ and Ψ seen above.

Let us notice that the equilateral triangle formed with the three asymptotes is centered at O.

If we call U_t, V_t, W_t the isotomic conjugates of the points at infinity of the three asymptotes (they are obviously on the Steiner circum-ellipse and on the cubic), the lines through the Steiner point and U_t, V_t, W_t meet the circumcircle again at three points U'_t, V'_t, W'_t which are on the asymptotes and are the vertices of an equilateral triangle. We can remark that the second intersections (other than U'_t, V'_t, W'_t) of the asymptotes with the circumcircle are the vertices of another equilateral triangle.

[7] [Paul Yiu] There are two lines that contain this point :

(1) The line through $\mathbf{t}K$, parallel to the Euler line.

(2) The line joining the centroid to the inversive image of the symmedian point in the circumcircle, point called X_{187} in [38, 39]

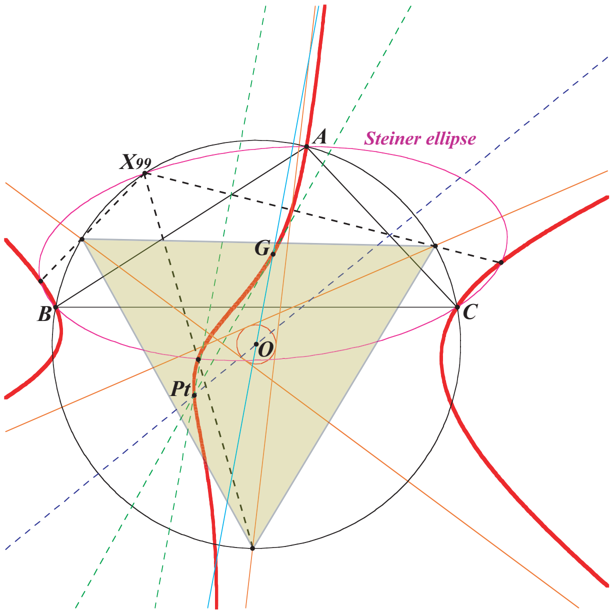

Figure 6.8: **K092** or $p\mathcal{K}_t$ the isotomic $p\mathcal{K}_{60}$

The points U_a, U_b, U_c we met above are on $p\mathcal{K}_t$. This means that $p\mathcal{K}_t$ is the member of $\mathcal{F}$ passing through G. According to theorem seen in §6.5, $p\mathcal{K}_t$ is the locus of pivots of all $p\mathcal{K}_{60}$ having the same asymptotic directions than $p\mathcal{K}_t$ itself. It can be seen that $p\mathcal{K}_t$ is the only $p\mathcal{K}_{60}$ having this particularity.

6.6.3 K037 the Tixier equilateral cubic

When we take P_o as pole of the isoconjugation, we find a very nice $p\mathcal{K}_{60}^{+}$ we shall call the Tixier equilateral cubic. See figure 6.9.

Its pivot is X_{30} and the three asymptotes concur at X_{476} which is a point on the circumcircle called Tixier point. (See [39] for details)

One of the asymptotes is the parallel at X_{476} to the Euler line and passes through X_{74}, the remaining two are easily obtained with 60° rotations about X_{476}.

Its equation is :

$$\sum_{\text{cyclic}} \left[(b^2-c^2)^2 + a^2(b^2+c^2-2a^2)\right] x\,(r_o\, y^2 - q_o\, z^2) = 0$$

The intersection with the asymptote parallel to the Euler line is the P_o-isoconjugate of X_{30} which is the isogonal of the midpoint of OX_{110}.

The nine common points with the Neuberg cubic are A, B, C, the feet of the parallels to the Euler line at those vertices and finally X_{30} which is triple and we can remark the two cubics share a common asymptote.

The common polar conic of X_{30} in these two cubics [8] is the rectangular hyperbola through the in/excenters and through X_5 with center X_{476}. This means that any parallel

[8] Its equation is :

$$\sum_{\text{cyclic}} (b^2-c^2)(4S_A^2 - b^2c^2)x^2 = 0$$

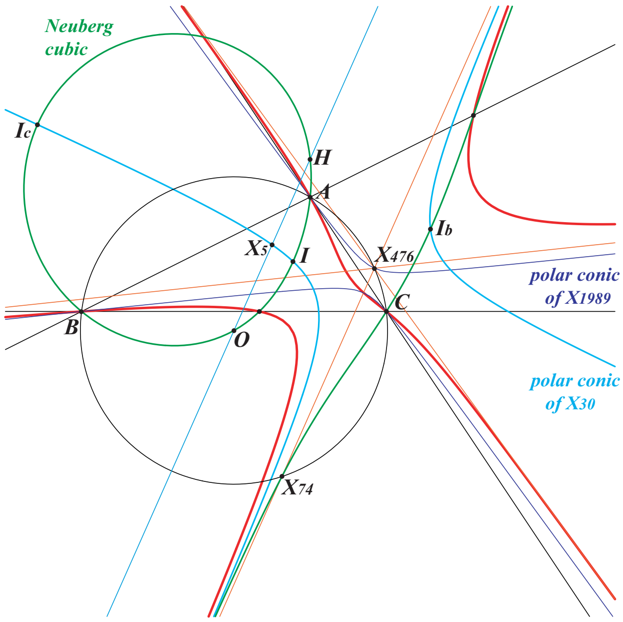

Figure 6.9: **K037** Tixier equilateral cubic and Neuberg cubic

to the Euler line meets the Neuberg cubic at M_1, M_2 and the Tixier cubic at N_1, N_2 such that the segments M_1M_2 and N_1N_2 have the same midpoint.

6.6.4 K046a and K046b the two Fermat cubics

Since there are only two points not on ABC sidelines – the Fermat points X_{13}, X_{14} – whose cevian lines make 60° angles with one another, there are only two non-degenerate $p\mathcal{K}_{60}^{++}$ which we call the Fermat cubics :

- their pivots are the anticomplements X_{616}, X_{617} [9] of the Fermat points.
- the poles are the points X_{396} and X_{395} [10] resp.
- the asymptotes concur on the cubic at the complements X_{618}, X_{619} of the Fermat points X_{13}, X_{14} respectively. These complements are therefore centers of each cubic and inflexion points. The inflexional tangents are the lines $X_{618}X_{396}$ and $X_{619}X_{395}$ respectively. The tangents at the points X_{13}, X_{616} for the first, at X_{14}, X_{617} for the second are all parallel to the Euler line.

  The asymptotes are the lines through the center of each cubic and the midpoints of the sidelines of ABC.

6.6.5 K104 parallel trilinear polars $p\mathcal{K}_{60}$

Following the remark in §1.4.2, it is easy to see that there exists only one $p\mathcal{K}_{60}$ locus of point M such that the trilinear polars of M and its Ω-isoconjugate M^* are parallel. See figure 6.11.

[9] These anticomplements are unusual points on the Neuberg cubic.

[10] X_{396} is the midpoint of X_{13}, X_{15} and X_{395} is the midpoint of X_{14}, X_{16}. Those two points are on the line GK.

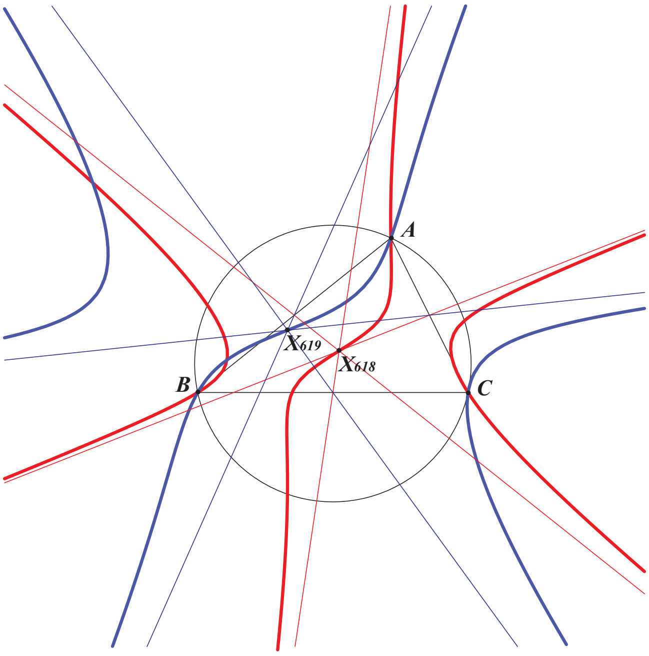

Figure 6.10: **K046a** and **K046b** the two Fermat cubics

The pivot and the pole are the isotomic of the point P_t we met in §6.6.2. An equation of the cubic is :

$$\sum_{\text{cyclic}} (4a^4 - 2a^2b^2 - 2b^4 - 2a^2c^2 + b^2c^2 - 2c^4)x^2(y - z) = 0$$

The triangle formed with the asymptotes is equilateral and centered at H.

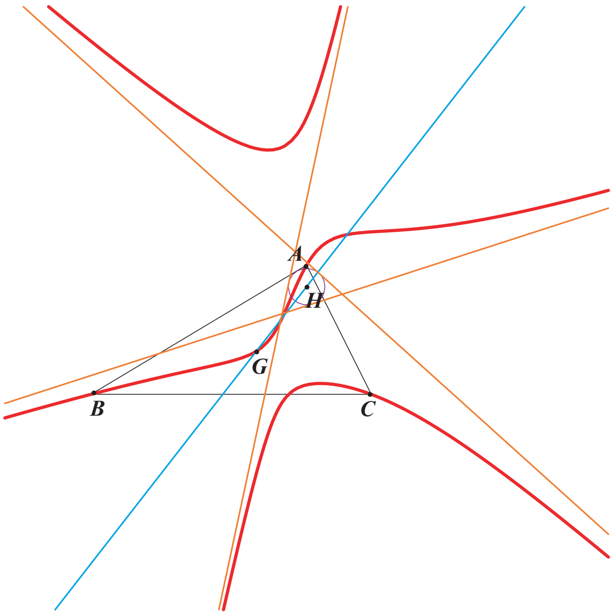

Figure 6.11: **K104** parallel trilinear polars $p\mathcal{K}_{60}$

6.7 A bicircular quartic

The locus of the intersection of the three asymptotes of all $p\mathcal{K}_{60}^{+}$ is a bicircular quartic $\mathcal{Q}_o$. See figure 6.12. Its equation is too complicated to be given here.

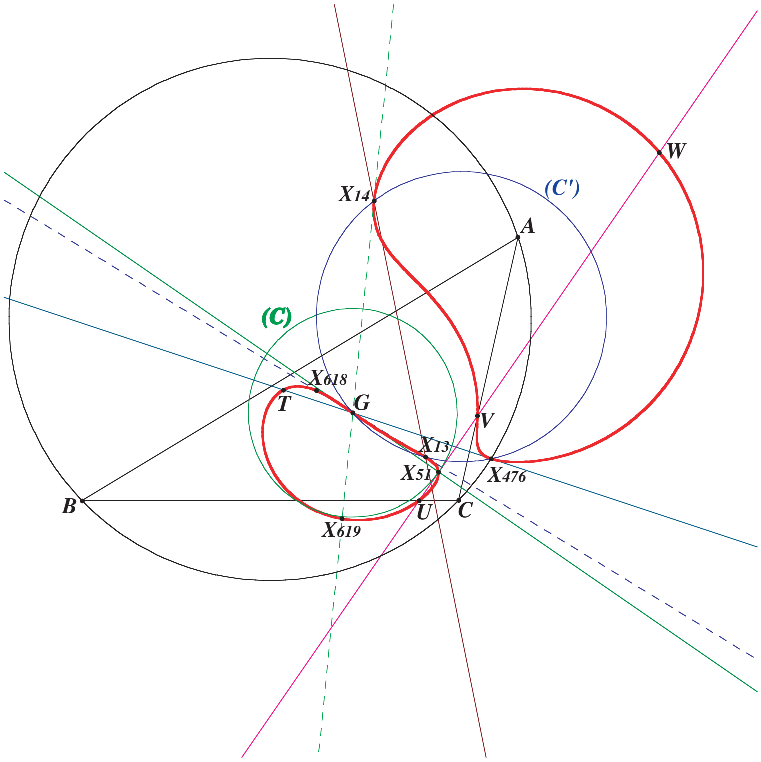

Figure 6.12: The bicircular quartic $\mathcal{Q}_o$ or **Q004**

$\mathcal{Q}_o$ passes through :

- the feet U, V, W of $\mathbb{P}(P_o)$.
- G, X_{51} (centroid of the orthic triangle) with an inflexional tangent at G which is the line GX_{51}.
- the Fermat points and their complements.
- the Tixier point X_{476} and its complement.

This quartic can be obtained by inverting a circular cubic in the following manner :

(C) is the circle centered at G through X_{51}. This circle is tangent at X_{51} to $\mathbb{P}(P_o)$. Let us call (k) the pivotal cubic with pivot the point at infinity of the parallel lines OK and GX_{51} which is isogonal with respect to the triangle formed by the inversive images U', V', W' [11] with respect to (C) of the points U, V, W. $\mathcal{Q}_o$ is the inversive image of (k) with respect to (C) which allows the construction.

Consequently, $\mathcal{Q}_o$ passes through the inversive images of the in/excenters of triangle $U'V'W'$.

This circular cubic (k) is very special in the sense that its singular focus lies on its asymptote. Hence, the line OK is perpendicular to one of the sides of the Morley triangle of $U'V'W'$.

[11] U', V', W' are three points on the circle with diameter GX_{51}.

Chapter 7

$n\mathcal{K}_{60}$ isocubics

7.1 Main theorem for $n\mathcal{K}_{60}$

For a given $\Omega(p : q : r)$-isoconjugation, let R_Ω be the point with coordinates

$$\left[p(4S_A^2 - b^2c^2) : q(4S_B^2 - c^2a^2) : r(4S_C^2 - a^2b^2)\right],$$

namely the $\Omega-$isoconjugate of $P_o = X_{1989}$.

There are only two possible situations :

- if $R_\Omega \neq G \iff \Omega \neq P_o$ then there is a pencil of $n\mathcal{K}_{60}$ all having the same asymptotic directions and their root on the line GR_Ω. Among them, there is one and only one $n\mathcal{K}_{60}^+$. [1]
- if $R_\Omega = G \iff \Omega = P_o$ then any point P of the plane is the root of a $n\mathcal{K}_{60}$.

  This $n\mathcal{K}_{60}$ is a $n\mathcal{K}_{60}^+$ if and only if P is on the line

  $$a^2(4S_A^2 - b^2c^2)x + b^2(4S_B^2 - c^2a^2)y + c^2(4S_C^2 - a^2b^2)z = 0$$

  which is the image of the circumcircle under the P_o-isoconjugation and, in fact, the perpendicular bisector of OH.

  In this case, the common point of the three asymptotes is on the circle Γ_o with radius $R/3$ (R circumcircle radius) which is homothetic to the circumcircle of ABC under $h(X_{125}, -1/3)$ or $h(H', 1/3)$ where X_{125} is the center of the Jerabek hyperbola and $H' = X_{265} = P_o$-isoconjugate of H.

Proof :

See §6.2 together with equation (2) in §1.3.

7.2 Isogonal $n\mathcal{K}_{60}$ cubics

In this paragraph, we take $\Omega = K$ and $R_\Omega = \left[a^2(4S_A^2 - b^2c^2)\right]$ which is X_{323} in [38, 39]. The common asymptotic directions of all cubics are the sidelines of the (first) Morley triangle. The root P is on the line GK.

[1] The pencil is generated by the two following cubics :

(1). one is degenerate into $\mathcal{L}^\infty$ and the circumconic which is its isoconjugate (this cubic is not a proper $n\mathcal{K}_{60}$)

(2). the other with root R_Ω and parameter 0.

When $P = G$ the cubic is decomposed into the union of the circumcircle and $\mathcal{L}^\infty$.
All those cubics meet the circumcircle at A, B, C and three other points which are the isogonal conjugates of the points at infinity of the Morley sidelines : those points are the vertices of an equilateral triangle called circumtangential triangle in [38]. See also [23].
For all those cubics, the triangle formed with the asymptotes is centered at G. If t is the real number such that $\overrightarrow{GP} = t\,\overrightarrow{GK}$, then the radius of the circumcircle of this triangle is :

$$\frac{a^2+b^2+c^2}{9R}\left|1-\frac{1}{t}\right|$$

7.2.1 The $\mathcal{K}_{jp}$ cubic or K024

When $P = K$, we obtain the only $n\mathcal{K}_{60}^{+}$ of the pencil denoted by $\mathcal{K}_{jp}$. See figure 7.1. This is a $n\mathcal{K}_0$ with equation :

$$\sum_{\text{cyclic}} a^2x(c^2y^2+b^2z^2) = 0 \quad \text{or} \quad \sum_{\text{cyclic}} (y+z)x^2/a^2 = 0$$

The asymptotes concur at G.

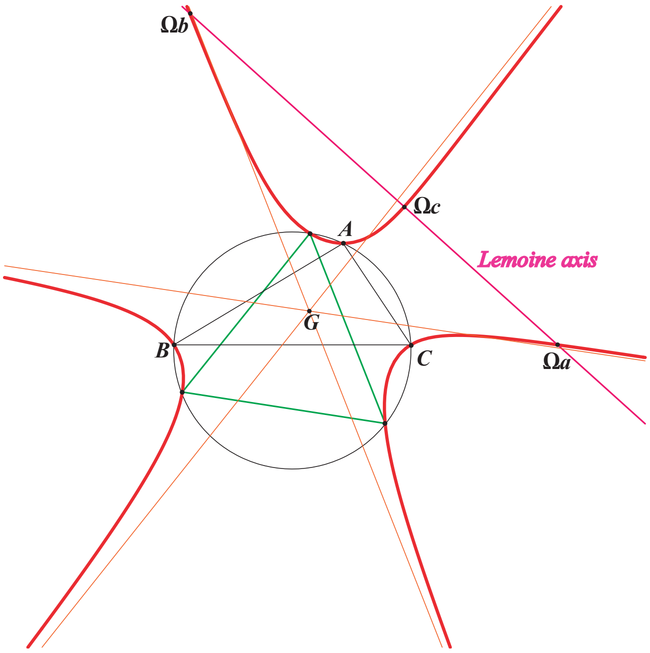

Figure 7.1: K024 or $\mathcal{K}_{jp}$

Among other things, this cubic is the locus of point M such that :

1. the circle with diameter $M\mathbf{g}M$ is orthogonal to the circumcircle.
2. the pedal circle of M is orthogonal to the nine-point circle.
3. the center of the conic $ABCM\mathbf{g}M$ is collinear with M and $\mathbf{g}M$ (together with the union of the six bisectors).
4. the pole of the line $M\mathbf{g}M$ in the conic $ABCM\mathbf{g}M$ lies on $\mathcal{L}^\infty$ (See §1.5.2) i.e. the tangents at M and $\mathbf{g}M$ to the conic $ABCM\mathbf{g}M$ are parallel. [2]

[2]This is true for every $n\mathcal{K}_0$ with pole = root.

5. the sum of line angles $(BC, AM) + (CA, BM) + (AB, CM) = 0$ (mod.π).
6. the Simson lines of M_a, M_b, M_c are concurrent, where M_a, M_b, M_c are the second intersections of the cevian lines of M with the circumcircle i.e. the vertices of the circumcevian triangle of M.
7. the lines $A\mathbf{g}M$ and M_bM_c (or equivalently $B\mathbf{g}M$ and M_cM_a, $C\mathbf{g}M$ and M_aM_b) are parallel.
8. the parallels at M_a, M_b, M_c to the sidelines BC, CA, AB respectively are concurrent.
9. the circumcevian triangles of M and $\mathbf{g}M$ are perspective (together with the union of the six bisectors, the circumcircle and $\mathcal{L}^\infty$).

This cubic passes through the centers of the three Apollonius circles, the tangents at those points being concurrent at G. The tangents at A, B, C are parallel to the sidelines of ABC.

7.2.2 K105 $n\mathcal{K}_{60}$ and perpendicular polar lines in the circumcircle

The locus of point M such that the polar lines (in the circumcircle) of M and $\mathbf{g}M$ are perpendicular is the isogonal $n\mathcal{K}_{60}$ noted **K105** with root the point $O^\perp = (\cos 2A; \cos 2B; \cos 2C)$ [3]. See figure 7.2

Its equation is :

$$(a^2 + b^2 + c^2)xyz + \sum_{\text{cyclic}} \cos 2A\; x(c^2y^2 + b^2z^2) = 0$$

One of the simplest geometric description of this cubic is the locus of point M such that the circle with diameter $M\mathbf{g}M$ passes through O.

It is also the locus of the intersections of a line ℓ through O with the rectangular hyperbola which is the isogonal image of the perpendicular at O to ℓ.

This cubic passes though O, H and $O_a^\perp, O_b^\perp, O_c^\perp$. The third point on OH is on the tangent at X_{110} to the circumcircle.

The third point on AH is $AH \cap OO_a^\perp$ and the third point on AO is $AO \cap HO_a^\perp$.

Its asymptotes form an equilateral triangle centered at G whose circumcircle has radius $2R/3$. This triangle is homothetic to the circumtangential triangle under $h_{H,2/3}$ and both triangles are homothetic to Morley triangle.

7.2.3 K085 an isogonal conico-pivotal $n\mathcal{K}_{60}$

In [24], we have met an isotomic non-pivotal cubic called "the Simson cubic" for which the line through a point M on the curve and $\mathbf{t}M$ envelopes an ellipse inscribed in the antimedial triangle $G_aG_bG_c$. We call this conic the pivotal conic of the cubic and say the cubic is a conico-pivotal cubic.

Similarly, we find an isogonal conico-pivotal $n\mathcal{K}_{60}$ with root at $[a(a^2 + ab + ac - 2bc)]$ (on the line GK) with equation :

$$-2abc(a^2 + b^2 + c^2)xyz + \sum_{\text{cyclic}} a(a^2 + ab + ac - 2bc)\; x(c^2y^2 + b^2z^2) = 0$$

[3] We call the point $O^\perp$ "orthocorrespondent" of O : the perpendicular lines at O to the lines OA, OB, OC meet the sidelines BC, CA, AB at three collinear points $O_a^\perp, O_b^\perp, O_c^\perp$ called the "orthotraces" of O and then $O^\perp$ is the tripole of the line through those three points. We call this line "orthotranversal" of O. $O^\perp$ is obviously on GK since $\cos 2A = 1 - 2\sin^2 A$.

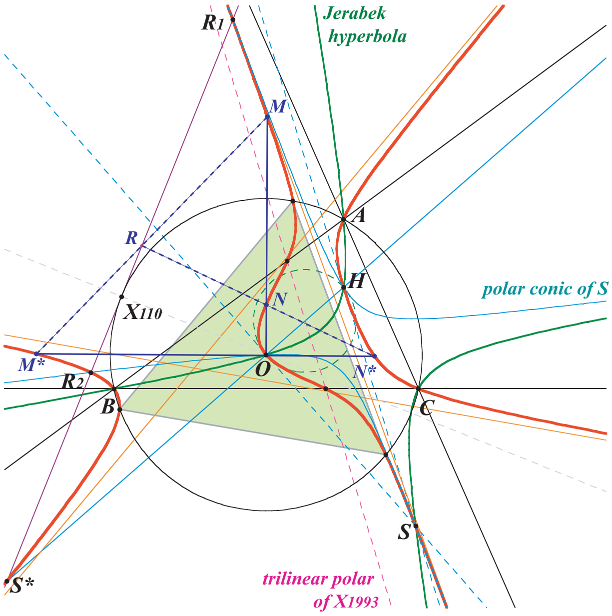

Figure 7.2: **K105** a $n\mathcal{K}_{60}$ related to perpendicular polar lines

It has a singularity at I (incenter) and any line through a point M on the curve and $\mathbf{g}M$ enveloppes a conic inscribed in the triangle $I_aI_bI_c$ (excenters). See figure 7.3.

This cubic is the locus of point M such that the circle with diameter $M\mathbf{g}M$ is orthogonal to the circle centered at O passing through I.

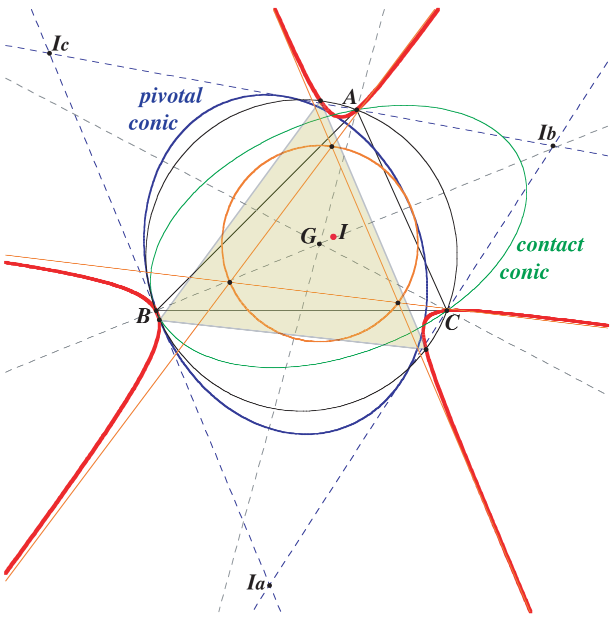

Figure 7.3: **K085** an isogonal conico-pivotal $n\mathcal{K}_{60}$

7.2.4 K098 the NPC pedal cubic

The locus of point M such that the pedal circle of M passes through a fixed point Q is an isogonal $n\mathcal{K}$ with root $Q^{\perp}$.

This cubic becomes a $n\mathcal{K}_{60}$ if and only if $Q = X_5$. It is denoted **K098**. See figure 7.4. Its root is, as usual, a point on the line GK. Here, this is $X_5^{\perp}$ with coordinates :

$$\left[a^2(a^4+b^4+c^4-2a^2b^2-2a^2c^2-b^2c^2)\right].$$

Its equation is :

$$\sum_{\text{cyclic}} a^2(a^4+b^4+c^4-2a^2b^2-2a^2c^2-b^2c^2)\,x\,(c^2y^2+b^2z^2)+a^2b^2c^2(a^2+b^2+c^2)\,xyz=0$$

The triangle formed with the asymptotes has circumradius $R/3$, center G.

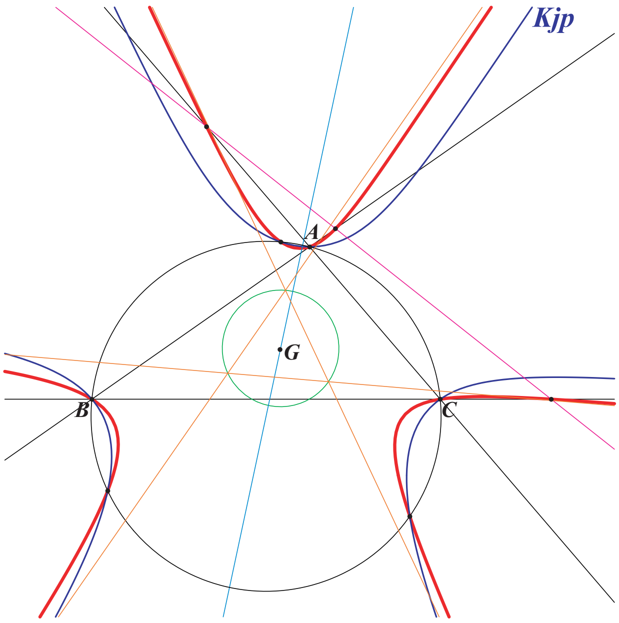

Figure 7.4: **K098** the NPC pedal cubic

7.3 Isotomic $n\mathcal{K}_{60}$ cubics

In this paragraph, we take $\Omega = G$ and $R_\Omega = \left[4S_A^2 - b^2c^2\right]$ which is $\mathbf{t}P_o$.

The common asymptotic directions of all cubics are again the sidelines of the (first) Morley triangle. The root P is on the line $G\mathbf{t}K$.

All those cubics meet the Steiner circum-ellipse at A, B, C and three other points T_a, T_b, T_c which are the isotomic conjugates of the points at infinity of the Morley sidelines : those points are the vertices of another equilateral triangle and are the intersections - other than the Steiner point X_{99} - of the Steiner circum-ellipse with the circle centered at $\mathbf{t}K$ through X_{99}. See [23] for details.

The center of the triangle formed with the asymptotes (center of the circle which is the poloconic of $\mathcal{L}^\infty$) is a point on the parallel to the Brocard line OK at the midpoint

X_{381} of GH [4].

Remark :

one of the cubics degenerates into the union of the Steiner circum-ellipse and $\mathcal{L}^\infty$ and is obtained when the root is G.

7.3.1 K094 the isotomic $n\mathcal{K}_{60}^{+}$

The only $n\mathcal{K}_{60}^{+}$ of the pencil is obtained with root

$$P = [b^2c^2(b^2 + c^2 - 5a^2)] \text{ [5]}$$

and its equation is :

$$2\left[(b^2 + c^2)(c^2 + a^2)(a^2 + b^2) - 8a^2b^2c^2\right] xyz + \sum_{\text{cyclic}} b^2c^2(b^2 + c^2 - 5a^2)x(y^2 + z^2) = 0$$

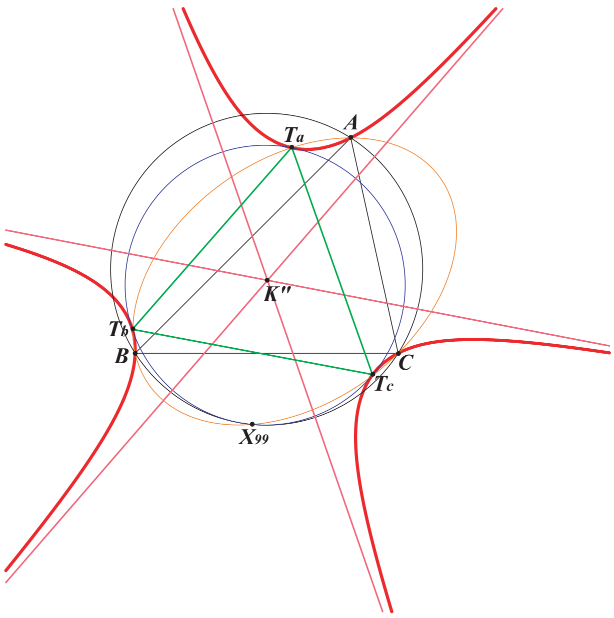

Figure 7.5: K094 the isotomic $n\mathcal{K}_{60}^{+}$

The three asymptotes are concurring at K'' symmetric of K about G [6].

The cubic intersects the sidelines of ABC at U, V, W which are on $\Delta_P = \mathbb{P}(P)$. This line is perpendicular to GK.

The tangents at A, B, C intersect the sidelines of ABC at A_1, B_1, C_1 which lie on $\Delta_{\mathbf{t}P} = \mathbb{P}(\mathbf{t}P)$. This line is perpendicular to OK.

[4] An equation of this line is :

$$\sum_{\text{cyclic}} (b^2 - c^2)(4a^2S_A + b^2c^2)\, x = 0$$

[5] This point is the isotomic of the isogonal of the symmetric of G about K which is $[b^2 + c^2 - 5a^2]$

[6] K'' is X_{599} in [38, 39]. Its coordinates are $(2b^2 + 2c^2 - a^2)$. It is the perspector of the Lemoine ellipse i.e. inscribed ellipse with foci G and K.

7.3.2 Other examples of isotomic $n\mathcal{K}_{60}$

- **K093** We find a very simple isotomic $n\mathcal{K}_{60}$ when the root is

$$[2a^2(b^2+c^2)-b^2c^2]$$

which is the symmetric of $\mathbf{t}K$ about G. See figure 7.6.

This is a $n\mathcal{K}_0$ whose equation is :

$$\sum_{\text{cyclic}} \left[2a^2(b^2+c^2)-b^2c^2\right] x\,(y^2+z^2)=0$$

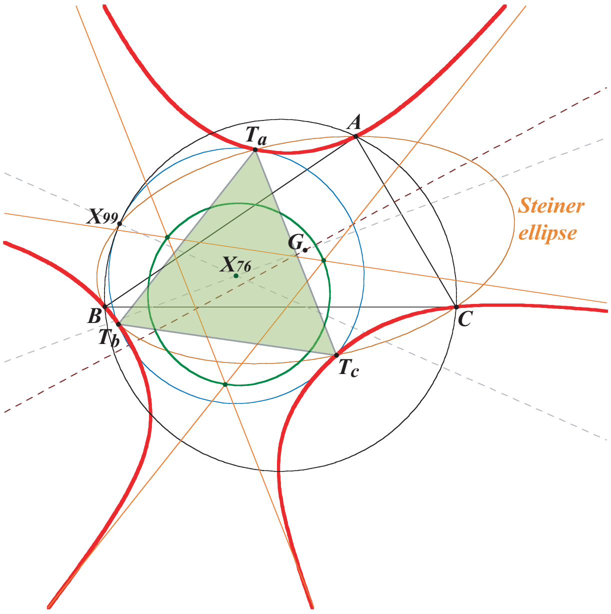

Figure 7.6: **K093** an isotomic $n\mathcal{K}_{0-60}$

- **K198** When the root is $\mathbf{t}K$, we find a remarkable isotomic $n\mathcal{K}_{60}$ with equation :

$$\sum_{\text{cyclic}} b^2c^2x\,(y^2+z^2)+2(b^2c^2+c^2a^2+a^2b^2)xyz=0$$

In this case, the circumcircle of the triangle formed with the asymptotes is centered at X_{381} (midpoint of GH) and has radius $2R$ (circumcircle radius). See figure 7.7.

7.3.3 K089 an isotomic conico-pivotal $n\mathcal{K}_{60}$

When we take the point $P=[4a^2(b^2+c^2)-5b^2c^2]$ [7] as root, we obtain another isotomic conico-pivotal cubic (see [24]) which is now a $n\mathcal{K}_{60}$ denoted **K089**. See figure 7.8.

Its equation is :

$$\sum_{\text{cyclic}} \left[4a^2(b^2+c^2)-5b^2c^2\right] x(y^2+z^2)-6\,(b^2c^2+c^2a^2+a^2b^2)\,xyz=0$$

[7] This point is the homothetic of $\mathbf{t}K$ under $h(G,-3)$.

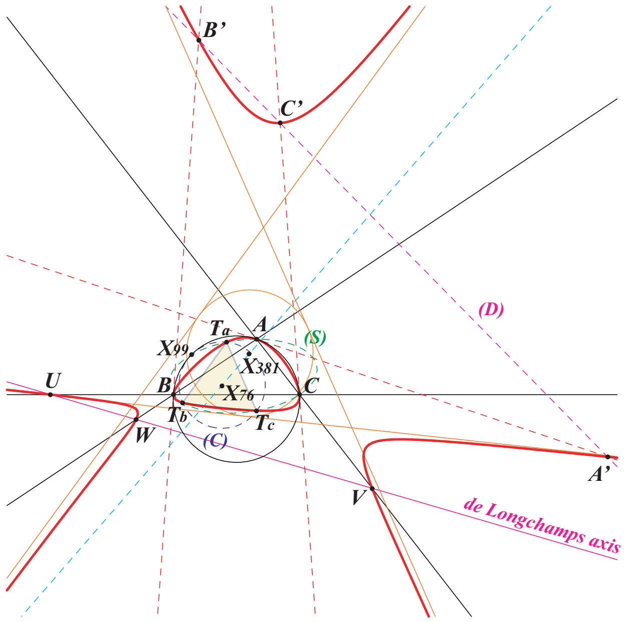

Figure 7.7: **K198** an isotomic $n\mathcal{K}_{60}$

This cubic has a singularity at G.

The line through M and $\mathbf{t}M$ envelopes the ellipse $\mathcal{E}$ inscribed in $G_aG_bG_c$ and also in the triangle whose sidelines join a vertex of ABC to the foot of the tripolar of P on the opposite side. Knowing six tangents, it is easy to draw $\mathcal{E}$. Its center is $[a^2(b^2+c^2)-8b^2c^2]$ on $G\mathbf{t}K$ [8].

The tangents at G are the tangents drawn through G to $\mathcal{E}$ (they can be real or imaginary).

It is fairly easy to draw the asymptotes : the tangent at T_a to the circle centered at $\mathbf{t}K$ is also tangent to $\mathcal{E}$ at t_a. The symmetric (which is on the cubic) of t_a about T_a has its isotomic conjugate on the tangent at T_a and on the asymptote parallel to T_bT_c.

7.4 $n\mathcal{K}_{60}^{++}$ isocubics

We have seen that neither isogonal nor isotomic conjugation give a $n\mathcal{K}_{60}^{++}$. It seems natural to seek such cubics for another Ω-isoconjugation. We have the following theorem :

7.4.1 Main theorem for $n\mathcal{K}_{60}^{++}$ isocubics

Each $n\mathcal{K}_{60}^{++}$ isocubic is obtained in the following manner :

Let M be a point on the circumcircle, M' its orthogonal projection on its Simson line, N such that $\overrightarrow{MN} = 4/3\ \overrightarrow{MM'}$ and Ω the barycentric product of N and $\mathbf{g}M$.

Then N is the center of the $n\mathcal{K}_{60}^{++}$ isocubic which is invariant under Ω-isoconjugation and which has the line MM' as asymptote.

The root can be obtained according to §3.3.2.

[8] This point is the homothetic of $\mathbf{t}K$ under $h(G, 3/2)$.

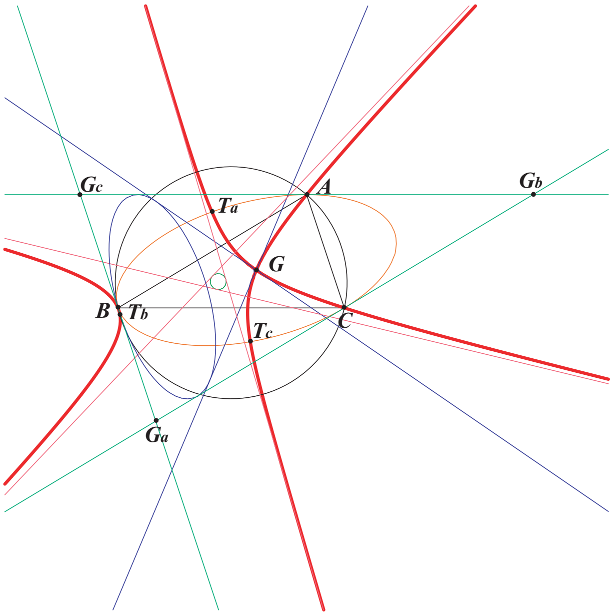

Figure 7.8: **K089** an isotomic conico-pivotal $n\mathcal{K}_{60}$

Remarks :

1. The locus of Ω is a conic $\mathcal{C}_1$ through P_o whose equation is :

$$\sum_{\text{cyclic}} (4S_A^2 - b^2c^2)x^2 + \left[a^2(a^2+b^2+c^2) - 2(b^2-c^2)^2\right] yz = 0$$

   Its center is the point $\left[a^2(5b^2+5c^2-7a^2) - 4(b^2-c^2)^2\right]$ on the line GK.

2. The locus of N is a bicircular circum-quartic $\mathcal{Q}_1$ easy to draw.

3. The asymptote MM' is a Simson line with respect to triangle $J_aJ_bJ_c$ (excenters). This means that there are three $n\mathcal{K}_{60}^{++}$ isocubics with given asymptotic directions. They can be obtained with three points M_1, M_2, M_3 vertices of an inscribed equilateral triangle in the circumcircle.

4. The cubic can degenerate into a line and a conic : that happens for example when the point M on the circumcircle is the second intersection of an internal bisector.

7.4.2 K139 the P_o-isoconjugate $n\mathcal{K}_{60}^{++}$ cubic

There is only one $n\mathcal{K}_{60}^{++}$ with pole $P_o = X_{1989}$ obtained when $M = X_{74}$. This is **K139**. See figure 7.9.

One of its asymptotes is parallel to the Euler line and the three asymptotes concur at the P_o-isoconjugate of X_{30} which is :

$$\left[\frac{1}{(4S_A^2 - b^2c^2)[(b^2-c^2)^2 + a^2(b^2+c^2-2a^2)]}\right].$$

Its equation is :

$$\sum_{\text{cyclic}} (b^2-c^2)E_a\, x\,(r_o y^2 + q_o z^2) - 6(a^2-b^2)(b^2-c^2)(c^2-a^2)xyz = 0$$

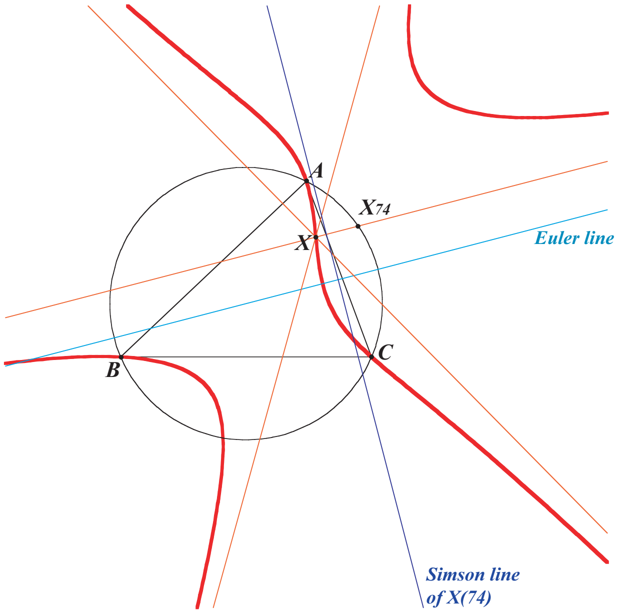

Figure 7.9: **K139** a $n\mathcal{K}_{60}^{++}$

with

$$E_a = 8S_A^4 - 4b^2c^2S_A^2 + 9a^2b^2c^2S_A - 4b^4c^4$$

and where E_b and E_c are found similarly. See §6.1 for p_o, q_o, r_o.

7.4.3 Other remarkable Ω-isoconjugate $n\mathcal{K}_{60}^{++}$ isocubics

We have been unable to find a reasonnably simple cubic with the most commonly used points on the circumcircle. One of the less complicated has center with first barycentric coordinate :

$$(a+b-2c)(a-2b+c)(a^5 - 2a^4b + 7a^3b^2 + 4a^2b^3 - 5ab^4 + b^5 - 2a^4c - 9a^3bc$$
$$-5a^2b^2c + b^4c + 7a^3c^2 - 5a^2bc^2 + 10ab^2c^2 - 2b^3c^2 + 4a^2c^3 - 2b^2c^3 - 5ac^4 + bc^4 + c^5)$$

and pole :

$$\left[(b-c)(3a^3 - 2a^2b - 4ab^2 + b^3 - 2a^2c + 7abc - 4ac^2 + c^3)\right]$$

One of its asymptote is parallel to the line IG.

7.5 A $\mathcal{K}_{60}^{+}$ summary

Ω being a point not on the sidelines of ABC, we can sum up the study in the following manner :

- if $\Omega \notin \mathcal{C}_o$, there is only one $\mathcal{K}_{60}^{+}$ with pole Ω and this cubic is non-pivotal. (this is the case of the isotomic conjugation)
- if $\Omega \in \mathcal{C}_o$ and $\Omega \neq P_o$, there are exactly two $\mathcal{K}_{60}^{+}$ with pole Ω :

  - one is pivotal with pivot on the Neuberg cubic and its asymptotes concurring on the quartic $\mathcal{Q}_o$. (This $\mathcal{K}_{60}^{+}$ is $\mathcal{K}_{60}^{++}$ if and only if the pivot is the anticomplement of one of the Fermat points)
  - the other is non-pivotal. (this is the case of the isogonal conjugation, the cubics being the McCay cubic and $\mathcal{K}_{jp}$)

- if $\Omega = P_o$, there are infinitely many $\mathcal{K}_{60}^{+}$ with pole Ω :
  - one only is pivotal (the Tixier cubic with pivot X_{30}).
  - the others are non-pivotal with their root on the perpendicular bisector of OH and their asymptotes concurring on the circle Γ_o.

Chapter 8

Conico-pivotal (unicursal) isocubics or $c\mathcal{K}$

8.1 Theorems and definitions

(See also [24])

- A $n\mathcal{K}$ with root $P = (u : v : w)$ is a **conico-pivotal isocubic** invariant under an isoconjugation if and only if it passes through one and only one of the fixed points $F = (f : g : h)$ of the isoconjugation : for any point M on the curve, the line through M and its isoconjugate M^* envelopes a conic we call the **pivotal-conic** of the cubic.

  In this case, the cubic has a singularity at F and is unicursal.

  A conico-pivotal isocubic will be denoted by $c\mathcal{K}$.

  An equation of this cubic is :

$$-2(ghu + hfv + fgw)xyz + \sum_{\text{cyclic}} ux(h^2y^2 + g^2z^2) = 0$$

  which rewrites as :

$$\sum_{\text{cyclic}} ux(hy - gz)^2 = 0$$

  showing that, for a given F, those cubics are in a net generated by the three degenerated cubics into a sideline of ABC and the corresponding cevian line of F counted twice.

  The pivotal-conic $\mathcal{C}$ is inscribed in the precevian triangle of F (formed with the three other fixed points of the isoconjugation) and has equation :

$$\sum_{\text{cyclic}} \left[(gw - hv)^2x^2 - 2\left(ghu^2 + 3fu(hv + gw) + f^2vw\right)yz\right] = 0$$

  and center : $[u(g + h - 2f) + f(v + w)]$.

- The three contacts of $c\mathcal{K}$ and $\mathcal{C}$ lie on a circum-conic called **contact-conic** with equation :

$$\sum_{\text{cyclic}} \left(2\,\frac{f}{u} + \frac{g}{v} + \frac{h}{w}\right)\frac{f}{x} = 0$$

its fourth common point (assuming that the conics are distinct, see §8.3 below) with $\mathcal{C}$ being :

$$\left[\left(2\,\frac{f}{u}+\frac{g}{v}+\frac{h}{w}\right)/(gw-hv)\right]$$

- The isoconjugate of this circum-conic is the polar line of F in the pivotal-conic $\mathcal{C}$. An equation of this line is :

$$\sum_{\text{cyclic}}\left(2\,\frac{f}{u}+\frac{g}{v}+\frac{h}{w}\right)\frac{x}{f}=0$$

  This line passes through the real inflexion point(s) which is (are) isoconjugate(s) of the real contact(s).

- The two tangents at F to the cubic are the tangents (real or imaginary) drawn from F to $\mathcal{C}$. [1]

- When the pivotal-conic is a circle, $c\mathcal{K}$ is called a **cyclo-pivotal isocubic**.

We have already met examples of conico-pivotal isocubics in §7.2.3 and §7.3.3.

The goal of this paragraph is to systematize the research of such cubics and realize their construction.

8.2 Construction of a conico-pivotal isocubic and its pivotal conic

Given F and P as above, we denote by $F_aF_bF_c$ the precevian triangle of F and by U, V, W the feet of $\mathbb{P}(P)$.

Remark : The following constructions can be realized with a ruler only.

8.2.1 Starting from the pivotal conic

- First construct the pivotal-conic $\mathcal{C}$ entirely determined by six tangents : the sidelines of $F_aF_bF_c$ and the lines AU, BV, CW. Let Q be its contact with CW (or AU, or BV) and Q' the harmonic conjugate of Q with respect to C and W.

  A variable tangent (T) at T to $\mathcal{C}$ meets CW at t and let t' be the harmonic conjugate of t with respect to C and W. The tangent (T) and Ft' meet at M on the cubic.

- FT and CW intersect at E. E' is the harmonic conjugate of E with respect to F and T. The tangent at M to the cubic is the harmonic conjugate of ME' with respect to the lines MC and MW.

- From M we can draw another tangent (T') to $\mathcal{C}$. The lines MT' and FT meet at N on the cubic and the harmonic conjugate of M with respect to N and T' is M^*, isoconjugate of M. Note that the point $FT' \cap NN^*$ also lies on the cubic.

[1] More precisely, when the two nodal tangents are real the cubic is said to be crunodal and has only one real inflexion point. On the other hand, when these tangents are imaginary the cubic is said to be acnodal and has three real collinear inflexion points.

8.2.2 Starting from an inscribed conic

Let γ be the inscribed conic with perspector the cevian product Q of F and P. [2] Q is the perspector of triangles ABC and $A'B'C'$ where $A' = PF_a \cap BC, B'$ and C' defined likewise. A variable tangent τ to γ meets the lines FU, FV, FW at Z_a, Z_b, Z_c. The perspector Z of triangles ABC and $Z_aZ_bZ_c$ is a point of the cubic. Now, let us construct $A'' = FA' \cap F_bF_c$ and B'', C'' similarly. The perspector R of triangles $A''B''C''$ and $F_aF_bF_c$ is the perspector (with respect to $F_aF_bF_c$) of the pivotal-conic.

8.2.3 With trilinear polars

Let S_a be the trilinear pole of the line through the points $AB \cap FV$ and $AC \cap FW$. The points S_b and S_c being defined likewise, S_a, S_b and S_c are collinear on the line s whose trilinear pole S is the perspector of the inscribed conic γ seen in paragraph above. For any point Q on s, the line $q = \mathbb{P}(Q)$ intersects FU, FV, FW at Q_a, Q_b, Q_c respectively. The lines AQ_a, BQ_b, CQ_c concur on the cubic at M and the line q envelopes $\mathcal{C}$.

8.2.4 $c\mathcal{K}$ with given node passing through a given point

Let $F = (f : g : h)$ be the fixed point of an isoconjugation and the node of a $c\mathcal{K}$ with root $P = (u : v : w)$.

Let M be a point not lying on a cevian line of F. This cubic passes through $M = (\alpha : \beta : \gamma)$ (and also its isoconjugate M^*) if and only if P lies on the line L with equation :

$$\sum_{\text{cyclic}} \alpha\,(h\beta - g\gamma)^2\,x = 0.$$

This line L passes through the trilinear poles of the lines FM and FM^* with first barycentric coordinates :

$$\frac{1}{h\beta - g\gamma} \quad \text{and} \quad \frac{f}{\alpha\,(h\beta - g\gamma)} \quad \text{respectively.}$$

It follows that there is a pencil of cubics $c\mathcal{K}$ with node F passing through M and M^*.

This pencil is generated by two decomposed cubics namely the union of the line FM (resp. FM^*) and the circum-conic which is its isoconjugate hence passing through F, M^* (resp. F, M).

8.3 Circum-conico-pivotal isocubics

- When $\mathcal{C}$ is a circum-conic, $c\mathcal{K}$ is called circum-conico-pivotal isocubic. A quick computation shows that the root must be F therefore, for a given isoconjugation, there are four circum-conico-pivotal isocubics.

  The equation of the cubic rewrites as :

$$-6\,fgh\,xyz + \sum_{\text{cyclic}} fx(h^2y^2 + g^2z^2) = 0 \iff \sum_{\text{cyclic}} fx(hy - gz)^2 = 0$$

[2] Q is called Ceva point in [39].

and the pivotal-conic is the circum-conic with perspector F (its equation is : $fyz + gzx + hxy = 0$) and center $G/F = [f(g+h-f)]$.

Hence, it is tritangent to the cubic at A, B, C and the cubic meets the sidelines of ABC at three points lying on $\mathbb{P}(F)$ which are inflexion points.

Let us remark that, in this case, the pivotal-conic and the contact-conic are identical.

This cubic can be seen as the locus of M intersection of a tangent to $\mathcal{C}$ with the trilinear polar of the point of tangency.

- **Examples**
  - **K228** when $F = I$, we obtain an isogonal $c\mathcal{K}$ with equation :

$$\sum_{\text{cyclic}} ax(cy-bz)^2 = 0$$

  - **K015** when $F = G$, we obtain the Tucker nodal cubic, an isotomic $c\mathcal{K}$ with equation :

$$\sum_{\text{cyclic}} x(y-z)^2 = 0$$

    and $\mathcal{C}$ is the Steiner circum-ellipse.
  - **K229** when $F = K$, we obtain a $c\mathcal{K}$ such that $\mathcal{C}$ is the circumcircle. Its equation is :

$$\sum_{\text{cyclic}} a^2x(c^2y-b^2z)^2 = 0$$

  - when $F = O$, $\mathcal{C}$ is the circum-ellipse with center K.
  - when $F = X_{523}$, we obtain a $c\mathcal{K}$ such that $\mathcal{C}$ is the Kiepert hyperbola.

    More generally, the pivotal-conic is a circum-rectangular hyperbola if and only if F lies on the orthic axis. The pole Ω of the isoconjugation lies on the inscribed ellipse with perspector X_{393} and center $[a^4 + (b^2-c^2)^2]$ intersection of the lines $X_2 - X_{39}$ and $X_4 - X_{32}$. This ellipse passes through X_{115} and has equation :

$$\sum_{\text{cyclic}} S_A^4\, x^2 - 2\, S_B^2 S_C^2\, y\, z = 0$$

  - when $F = X_{115}$, we obtain a $c\mathcal{K}$ such that $\mathcal{C}$ is the parabola with equation :

$$\sum_{\text{cyclic}} (b^2-c^2)^2\, yz = 0$$

    i.e. the isogonal conjugate of the tangent at X_{110} to the circumcircle [3].

    More generally, it is easy to see that the pivotal-conic is a circum-parabola if and only if F lies on the inscribed Steiner ellipse.

[3]This parabola passes through $X_{476}, X_{523}, X_{685}, X_{850}, X_{892}$.

8.4 Cyclo-pivotal isocubics

It is clear that there are four inscribed circles in the precevian triangle of F (in/excircles) therefore, for a given isoconjugation with four real fixed points, there are 16 cyclo-pivotal isocubics, sharing four by four the same pivotal-circle.

We have already met such a cubic in 8.3 with $F = K$ where $\mathcal{C}$ is the circumcircle.

8.4.1 Isogonal cyclo-pivotal isocubics

When $F = I$ (incenter), the precevian triangle of I is $I_aI_bI_c$ (excenters) and we can find four isogonal cyclo-pivotal isocubics with circles centered at X_{164} and harmonic associates.

8.4.2 Isotomic cyclo-pivotal isocubics

Taking G as fixed point of isotomic conjugation, we find four cyclo-pivotal isocubics with circles centered at the Nagel points passing through the Feuerbach points of the antimedial triangle. One of them is **K090** with equation :

$$\sum_{\text{cyclic}} (b + c - 3a)x(y^2 + z^2) + 2(a + b + c)xyz = 0$$

with root $X_{145} = \mathbf{a}X_8$ (X_8 = Nagel point), $\mathcal{C}$ being the circle centered at X_8 passing through X_{100}= $\mathbf{a}X_{11}$ (X_{11} = Feuerbach point) i.e. the incircle of the antimedial triangle. See figure 8.1.

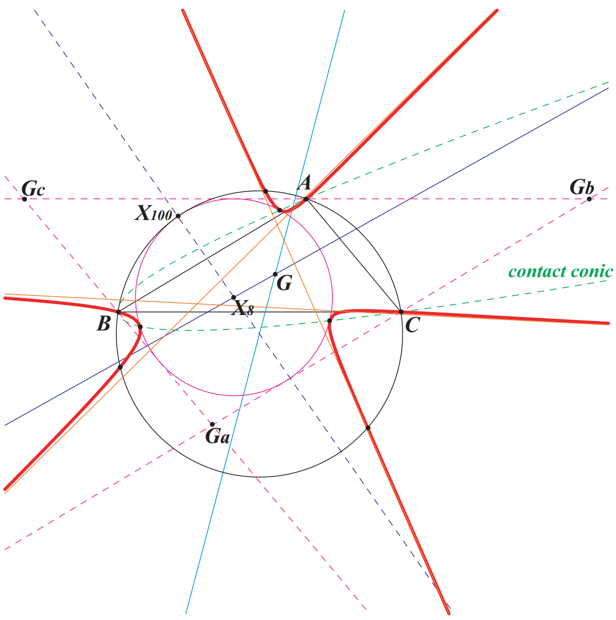

Figure 8.1: **K090** an example of isotomic cyclo-pivotal isocubic

8.5 Circular conico-pivotal isocubics

We find two very different situations according to the fact that the isoconjugation is or is not isogonal conjugation.

8.5.1 Circular isogonal conico-pivotal isocubics

A computation shows that, when we take $F = I$ (or one of the excenters), there is a pencil of circular isogonal conico-pivotal isocubics which are all strophoids as seen in §4.3.2.

The pivotal conic is a parabola.

8.5.2 Circular non-isogonal conico-pivotal isocubics

We now suppose that the conjugation is not isogonal conjugation.

Theorem :

For any point $F(f, g, h)$ not on ABC sidelines, there is one and only one circular non-isogonal conico-pivotal isocubic with singularity at F invariant under the isoconjugation having F as fixed point. Its root is :

$$R = \left[(a^2gh + b^2hf + c^2fg)^2 - b^2c^2f^2\,(f + g + h)^2\right]$$

clearly on the line through G and the isoconjugate of K.

an example : K088

Let us take $F = G$. There is one and only one circular isotomic conico-pivotal isocubic with node at G. This is **K088**. See figure 8.2. Its root is :

$$\left[(a^2 + b^2 + c^2)^2 - 9\,b^2c^2\right]$$

and its equation is :

$$\sum_{\text{cyclic}} \left[(a^2 + b^2 + c^2)^2 - 9b^2c^2\right] x(y^2 + z^2) - 6(a^4 + b^4 + c^4 - b^2c^2 - c^2a^2 - a^2b^2)xyz = 0$$

The pivotal conic is inscribed in the antimedial triangle and is centered at :

$$\left[(b^2 + c^2 - 2a^2)(2b^2 + 2c^2 - a^2)\right]$$

8.6 Equilateral conico-pivotal isocubics or $c\mathcal{K}_{60}$

In paragraphs §7.2.3 and §7.3.3 we have met two $c\mathcal{K}_{60}$, one is isogonal and the other isotomic.

More generally, for any isoconjugation with one fixed point $F = (f : g : h)$, there is one and only one $c\mathcal{K}_{60}$ with singularity at F. Its root is too complicated to be given here.

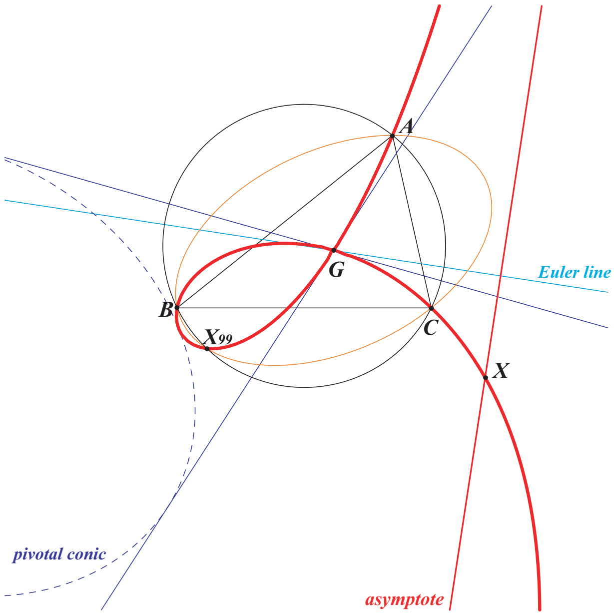

Figure 8.2: **K088** an example of circular isotomic conico-pivotal isocubic

This $c\mathcal{K}_{60}$ becomes a $c\mathcal{K}_{60}^{+}$ when F lies on a tricircular sextic with singularities at A, B, C passing through only one known center, namely $X_{80} = \mathbf{gi}I$. The corresponding $c\mathcal{K}_{60}^{+}$ is **K230** with root :

$$E_{596} = \left[\frac{b + c - 2a}{a(b + c - a)(b^2 + c^2 - a^2 - bc)} \right]$$

See figure 8.3.

The asymptotes concur at E_{597} a point lying on $X_{36}X_{80}$, the parallel at X_{80} to the line IH.

The tangents at X_{80} are perpendicular.

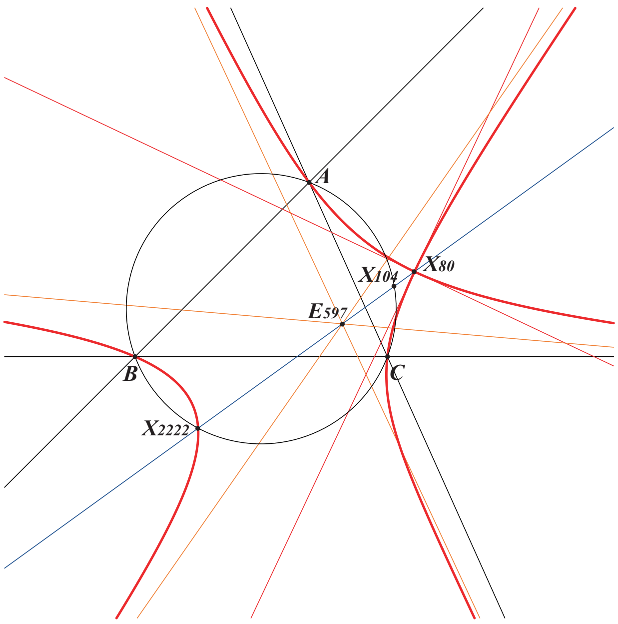

Figure 8.3: **K230** a $c\mathcal{K}_{60}^{+}$, an example of equilateral conico-pivotal isocubic

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