Two Related Transformations and Associated Cubics

Bernard Gibert

Created : April 1, 2014
Last update : April 6, 2014

Abstract

We study a transformation closely related with the orthocorrespondence and consequently a family of circum-cubics similar to orthopivotal cubics. All these cubics contain the four $Ix-$anticevian points.

1 The two transformations Φ and Ψ

1.1 Definition

The orthocorrespondence Φ is defined and studied in [6]. We simply recall that it maps a point $M = x : y : z$ to its orthocorrespondent $M^{\perp}$ with barycentric coordinates

$$x\,(-S_A x + S_B y + S_C z) + a^2\,yz : \cdots : \cdots . \tag{1}$$

Now, let us consider the transformation Ψ that maps M to its isogonal conjugate $M^{\bullet}$ with regard to the anticevian triangle of M. A computation easily gives the coordinates of $M^{\bullet}$ namely

$$x\,(-S_A x + S_B y + S_C z) - a^2\,yz : \cdots : \cdots . \tag{2}$$

These coordinates are surprisingly almost identical. Furthermore, by addition and substraction, we find that these two points lie on a same line that contains the isogonal conjugate M^* of M (with respect to the usual reference triangle ABC) and also the cevian quotient H/M (the perspector of the orthic triangle $H_aH_bH_c$ and the anticevian triangle of M also called $H-$Ceva conjugate of M). Indeed, $M^* = a^2\,yz : \cdots : \cdots$ and $H/M = x\,(-S_A x + S_B y + S_C z) : \cdots : \cdots$.

The points $M^{\bullet}$ and $M^{\perp}$ are clearly harmonic conjugates with respect to M^* and H/M.

The line passing through these four points does not contain M itself unless it lies on the Euler-Morley quintic Q003, a curve with many interesting properties.

1.2 Properties of the transformation Ψ

Singular points

When the coordinates of $M^{\bullet}$ are equated to zero, we obtain the equations of three conics σ_1, σ_2, σ_3 which generally have no common points.

σ_1 contains B, C, H_b, H_c and its tangents at B, C pass through the midpoint of the altitude AH.

σ_2 and σ_3 meet at A, H_a and two imaginary conjugate points but none of these points lies on σ_1. See figure 1.

It follows that Ψ has no singular point hence it always transforms a curve of degree n into a curve of degree $2n$.

Fixed points

When we express that M and $M^{\bullet}$ coincide, we find three nodal circum-cubics with nodes at A, B, C hence these three latter points are fixed points.

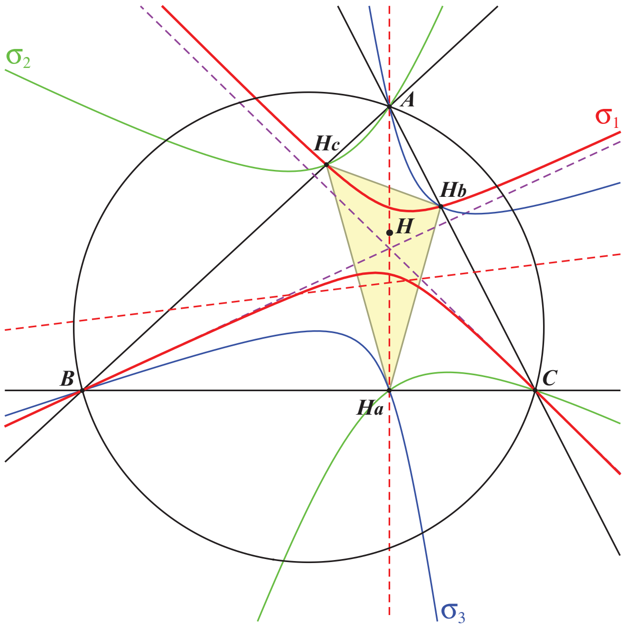

Figure 1: The conics σ_1, σ_2, σ_3

On the other hand, M (distinct of A, B, C) and $M^\bullet$ coincide if and only if M is an in/excenter of its anticevian triangle i.e. M is one of the four $Ix-$anticevian points of Table 23. These points lie on the rectangular hyperbola passing through X_5, X_6, X_{52}, X_{195}, X_{265}, X_{382}, X_{2574}, X_{2575}.

From this, we conclude that Ψ has seven fixed points which are the common points of several known cubics such as K005, K049, K060, etc. See §3.1 below.

1.3 Images of some triangle centers under Ψ

Angel Montesdeoca has kindly provided a list of ETC pairs (until X_{5550}) of the form $(X, \Psi(X))$:

(X_1, X_{40}), (X_2, X_{69}), (X_4, X_{20}), (X_6, X_{22}),
(X_{30}, X_{146}), (X_{511}, X_{147}), (X_{512}, X_{148}), (X_{513}, X_{149}), (X_{514}, X_{150}),
(X_{515}, X_{151}), (X_{516}, X_{152}), (X_{517}, X_{153}), (X_{523}, X_{3448}).

Surprisingly, the crop was poor and most of the points are images of points X at infinity. See below.

1.4 Images of lines under Ψ

Ψ transforms any line $\mathcal{L}$ into a conic $\mathcal{C}$ and, obviously, any line passing through one (or two) fixed point(s) into a conic passing through this (these) same fixed point(s).

Image of the line at infinity $\mathcal{L}_\infty$

It is easy to verify that Ψ swaps the circular points at infinity hence Ψ transforms $\mathcal{L}_\infty$ into a circle which turns out to be the circle with center H, radius $2R$ i.e. the anticomplement of the circumcircle.

1.5 The inverse transformation Ψ^{-1} of Ψ

The points $\Psi(M)$ and P coincide if and only if M is the intersection of three conics γ_A, γ_B, γ_C belonging to a same pencil hence having four common points but these points are not necessarily real.

The conic γ_A can be constructed easily since it contains five known points namely :
– A, H_a, the trace U on BC of the trilinear polar $\mathcal{L}(P)$ of P,
– the projections A_b, A_c on AB, AC of the intersection P_a of AP and BC.

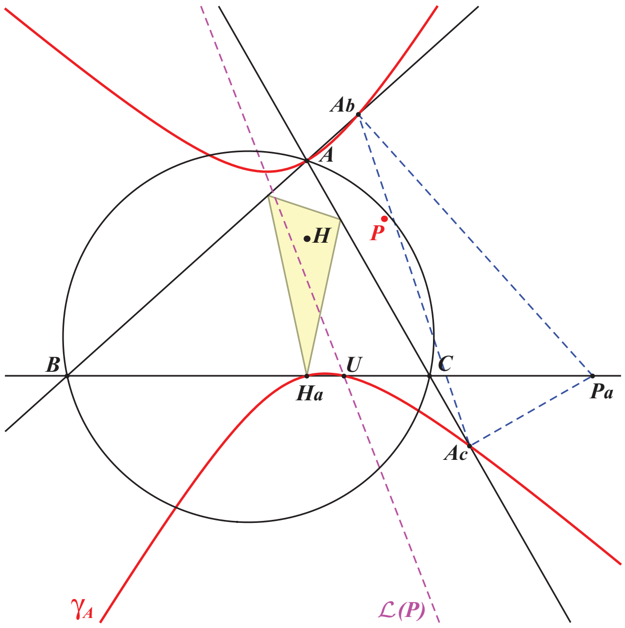

Figure 2: Construction of γ_A

2 Related cubic curves

2.1 Definitions and equations

Let $P = p : q : r$ be a given point. Recall that the orthopivotal cubic $\mathcal{O}(P)$ is the locus of M such that P, M and its orthocorrespondent $M^\perp = \Phi(M)$ are collinear. Any orthopivotal cubic $\mathcal{O}(P)$ passes through seven fixed points namely A, B, C, the circular points at infinity and the Fermat points X_{13}, X_{14}.

We define the "anticevian isogonal" cubic $\mathcal{A}(P)$ similarly : $\mathcal{A}(P)$ is the locus of M such that P, M and $M^\bullet = \Psi(M)$ are collinear.

The barycentric equation of the orthopivotal cubic $\mathcal{O}(P)$ is

$$\sum_{\text{cyclic}} 2p\,(S_B y - S_C z)yz - \sum_{\text{cyclic}} a^2(ry - qz)yz = 0 \tag{3}$$

and that of the anticevian isogonal cubic $\mathcal{A}(P)$ is

$$\sum_{\text{cyclic}} 2p\,(S_B y - S_C z)yz + \sum_{\text{cyclic}} a^2(ry - qz)yz = 0 \tag{4}$$

where

$$\sum_{\text{cyclic}} 2p\,(S_B y - S_C z)yz = 0 \tag{5}$$

is the equation of $p\mathcal{K}(H \times P, P)$, the pivotal cubic with pivot H and isopivot P, and

$$\sum_{\text{cyclic}} a^2(ry - qz)yz = 0 \tag{6}$$

is the equation of $p\mathcal{K}(X_6, P)$, the isogonal pivotal cubic with pivot P.

From these equations, we can see that $\mathcal{O}(P)$ and $\mathcal{A}(P)$ belong to a same pencil of cubics generated by the two pivotal cubics above which also contains a third (rather complicated) pivotal cubic.

All these cubics $\mathcal{A}(P)$ are therefore of type $\mathcal{K}_0$ i.e. cubics without term in xyz sometimes called "apolar cubics". See figure 3.

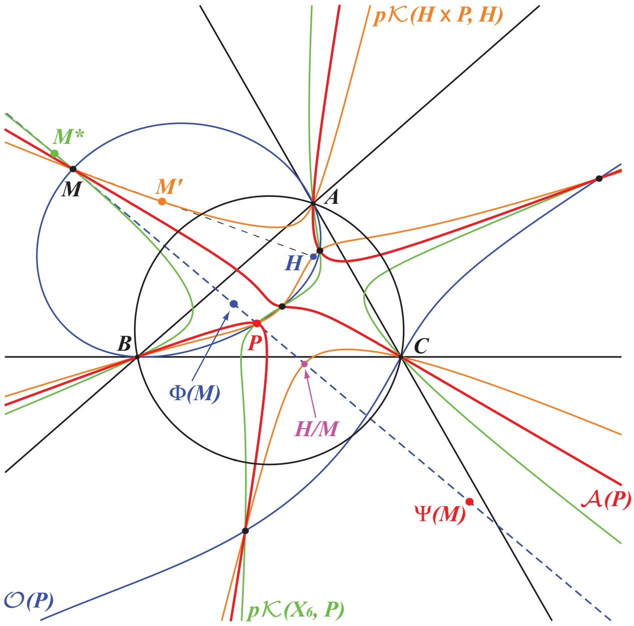

Figure 3: The pencil $\mathcal{A}(P)$, $\mathcal{O}(P)$, $p\mathcal{K}(X_6, P)$, $p\mathcal{K}(H \times P, P)$

Note that, when a point M is common to the four cubics (as in figure 3), then six points must be collinear namely P, M, M^*, H/M, $M^\bullet$ and $M^\perp$. Hence, M must be a point on the Euler-Morley quintic Q003. In other words, the four cubics of the pencil pass through A, B, C, P and five other points on Q003.

2.2 Immediate properties of $\mathcal{A}(P)$

From the definition above, we directly obtain that $\mathcal{A}(P)$ must contain the seven fixed points of Ψ. In other words, $\mathcal{A}(P)$ is a circum-cubic that passes through the four $Ix-$anticevian points.

$\mathcal{A}(P)$ must also contain P and the four (real or not) pre-images of P i.e. the points M such that $\Psi(M) = P$.

$\mathcal{A}(P)$ meets the sidelines of ABC again at three points U, V, W. This point U is the intersection of BC with the line passing through P and the reflection A' of A about BC with coordinates $0 : 2S_C\,p + a^2q : 2S_B\,p + a^2r$.

U is undefined when $P = A'$ since $\mathcal{A}(A')$ splits into the line BC and a conic.

2.3 Construction of $\mathcal{A}(P)$

Let us consider the pencil $\mathcal{F}$ of conics containing γ_A, γ_B, γ_C as in §1.5 and let P be a fixed point.

If L_P is a variable line passing through P, let C_P be the conic of the pencil $\mathcal{F}$ that contains the anticomplement of the isogonal conjugate of the infinite point of L_P. Recall that this latter point lies on the circle $C(X_4, 2R)$.

L_P and C_P meet at two points on $\mathcal{A}(P)$.

2.4 Intersection of $\mathcal{A}(P)$ with the line at infinity

The equation of $\mathcal{A}(P)$ rewrites under the form

$$\sum_{\text{cyclic}} a^2[(p+q)y-(p+r)z]yz = (x+y+z)\sum_{\text{cyclic}}\left[p\,(b^2-c^2)+a^2(q-r)\right]yz \qquad (7)$$

showing that $\mathcal{A}(P)$ meets the line at infinity at the same points as $p\mathcal{K}(X_6, cP)$, the pivotal isogonal cubic with pivot the complement cP of P and whose equation is the left hand member of (7).

It follows that $\mathcal{A}(P)$ and $p\mathcal{K}(X_6, cP)$ meet at six other finite points which lie on the circum-conic Γ_P with equation

$$\sum_{\text{cyclic}}[p\,(b^2-c^2)+a^2(q-r)]yz = 0,$$

passing through the centroid G of ABC. The perspector of this conic is the infinite point of the polar of P in the circumcircle as far as P is not O. If $P = O$, the two cubics $\mathcal{A}(P)$ and $p\mathcal{K}(X_6, cP)$ coincide with the Napoleon cubic K005. See figure 4.

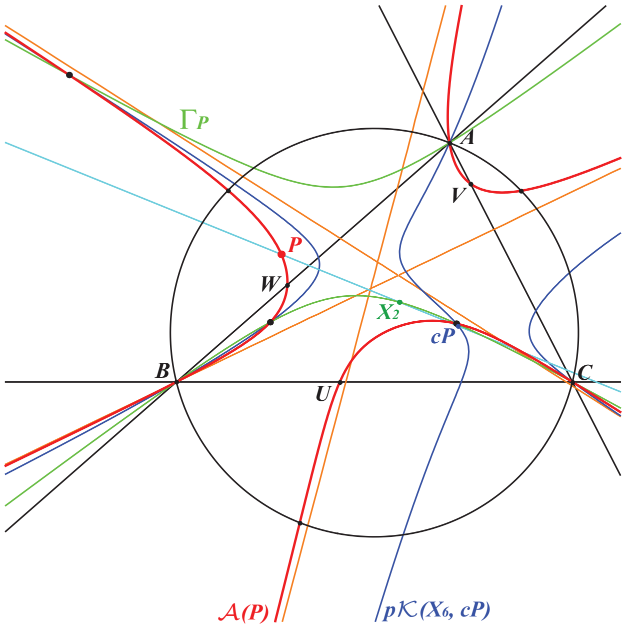

Figure 4: Intersection of $\mathcal{A}(P)$ with the line at infinity

2.5 Intersection of $\mathcal{A}(P)$ with the circumcircle of ABC

The equation of $\mathcal{A}(P)$ rewrites under the form

$$\begin{aligned}(a^2yz+b^2zx+c^2xy)\sum_{\text{cyclic}} b^2c^2[p\,(b^2-c^2)+a^2(q-r)]\,x \\ +\sum_{\text{cyclic}} a^2[2b^2c^2\,p-c^2(c^2-a^2)\,q+b^2(a^2-b^2)\,r]\,x\,(c^2y^2-b^2z^2)=0\end{aligned} \qquad (8)$$

showing that $\mathcal{A}(P)$ meets the circumcircle $(\mathcal{O})$ at the same points as the pivotal isogonal cubic $p\mathcal{K}(X_6, Q)$ with pivot

$$Q = a^2[2b^2c^2\,p-c^2(c^2-a^2)\,q+b^2(a^2-b^2)\,r] : \cdots : \cdots .$$

If G_p is the centroid of the pedal triangle of P and if ccP is the complement of the complement of P then Q is the homothetic of G_p under $h(ccP, 3)$.

It follows that $\mathcal{A}(P)$ and $p\mathcal{K}(X_6, Q)$ meet at three other (not necessarily all real) points which lie on the line Λ_P with equation

$$\sum_{\text{cyclic}} b^2c^2(a^2q - a^2r + b^2p - c^2p)\, x = 0,$$

which passes through X_6. The trilinear pole of Λ_P is the isogonal conjugate of the perspector of Γ_P. Here again, P must be different of O. See figure 5.

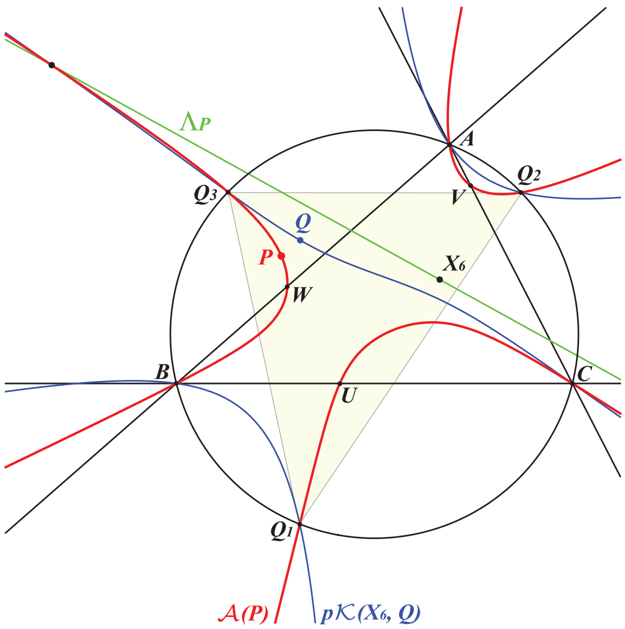

Figure 5: Intersection of $\mathcal{A}(P)$ with the circumcircle of ABC

3 Special cubics $\mathcal{A}(P)$

3.1 Pivotal cubics $\mathcal{A}(P)$

$\mathcal{A}(P)$ is a pivotal cubic (a $p\mathcal{K}$) if and only if the tangents at A, B, C concur (at the isopivot) or, equivalently, the third points U, V, W on the sidelines of ABC form a triangle perspective with ABC (at the pivot).

Recall that U is the intersection of BC with the line passing through P and the reflection A' of A about BC hence the line AU contains the reflection P_a of P about BC. The points P_b, P_c are defined likewise and then ABC and UVW are perspective if and only if ABC and $P_aP_bP_c$ are perspective therefore if and only if P lies on the Neuberg cubic K001.

Proposition 1 *$\mathcal{A}(P)$ is a pivotal cubic if and only if P lies on the Neuberg cubic* K001.

In this case,
– the pole lies on K095,
– the pivot lies on K060 which is $\mathcal{O}(X_5)$,
– the isopivot lies on the Napoleon cubic K005.
All these cubics contain the seven fixed points of Ψ as said above.

3.2 Non-pivotal cubics $\mathcal{A}(P)$

$\mathcal{A}(P)$ is a non-pivotal cubic (a $n\mathcal{K}_0$) if and only if the points U, V, W are collinear on a line whose trilinear pole is the root of the cubic.

Proposition 2 *$\mathcal{A}(P)$ is a non-pivotal cubic if and only if P lies on a non-pivotal isogonal cubic with root X_5.*

Unfortunately, this latter cubic does not contain any center of the current edition of ETC.

3.3 Cubics $\mathcal{A}(P)$ with concurring asymptotes

$\mathcal{A}(P)$ is a cubic with three real concurring asymptotes if and only if P lies on the stelloid Σ with equation

$$\sum_{\text{cyclic}} 2a^2 S_A \, x(c^2y^2 - b^2z^2) = (x+y+z)\, F(x,y,z) \tag{9}$$

where

$$F(x,y,z) = \sum_{\text{cyclic}} (b^2-c^2)[2S_A^2 x^2 - (a^2(8S_A - a^2) + (b^2-c^2)^2)yz].$$

The left hand member of (9) represents the McCay cubic K003 and $F(x,y,z) = 0$ is the equation of a rectangular hyperbola passing through H and tangent at H to the McCay cubic. Hence these two curves have four other common points S_1, S_2, S_3, S_4.

Σ has its asymptotes parallel to those of the McCay cubic and concurring at the reflection of X_{51} about H. It contains H, X_{550}, X_{3146} giving the cubics K049, K123, K127. See figure 6.

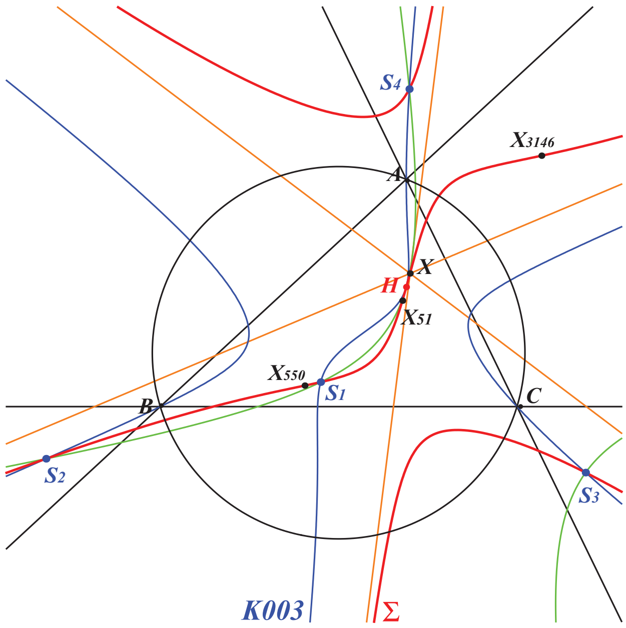

Figure 6: The stelloid Σ

3.4 Cubics $\mathcal{A}(P)$ with P on the Euler line

With P on the Euler line, the cubics $\mathcal{A}(P)$ form a pencil and contain X_4 and X_5. These are the cubics $\mathcal{D}(k)$ in [4].

References

[1] Ehrmann J.P. and Gibert B., *Special Isocubics in the Triangle Plane*, available at
http://bernard.gibert.pagesperso-orange.fr

[2] Gibert B., *Cubics in the Triangle Plane*, available at
http://bernard.gibert.pagesperso-orange.fr

[3] Gibert B., *Pseudo-Pivotal Cubics and Poristic Triangles*, available at
http://bernard.gibert.pagesperso-orange.fr

[4] Gibert B., *On two Remarkable Pencils of Cubics of the Triangle Plane*, available at
http://bernard.gibert.pagesperso-orange.fr

[5] Gibert B., *Orthocorrespondence and Orthopivotal Cubics*
Forum Geometricorum, Volume 3 (2003) 1–27.

[6] Gibert B., *How Pivotal Isocubics Intersect the Circumcircle*
Forum Geometricorum, Volume 7 (2007) 211–229.

[7] Kimberling C., *Triangle Centers and Central Triangles*, Congressus Numerantium, 129 (1998) 1–295.

[8] Kimberling C., *Encyclopedia of Triangle Centers*, 2000-2014
http://www2.evansville.edu/ck6/encyclopedia
http://faculty.evansville.edu/ck6/encyclopedia/ETC.html